Numerical Implementation of Hull-White Interest Rate Model: Hull-White Tree vs Finite Differences

Artur Sepp
Mail: artursepp@hotmail.com, Web: www.hot.ee/seppar
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Abstract

We implement the finite-difference (FD) solver and the Hull-White (HW) tree for numerical treatment of the pricing problem under the Hull-White interest rate model. We find that the FD solver is superior to the HW tree.

Contents

1 The Hull-White Interest Rate Model

We consider a simplified version of the Hull-White extension of the Vasicek model. In this specification, the $\mathbb{Q}$-dynamics of the short rate are given by

$$dr(t) = (\theta(t) - \kappa r(t))dt + \sigma dW(t) \tag{1.1}$$

where κ and σ are constant and $\theta(t)$ is a deterministic function of time.

The spot rate obeys the normal distribution. The parameter $\theta(t)$ is chosen with the purpose to fit the theoretical bond prices to the yield curve observed on the market. The parameter σ determines the overall level of volatility. The reversion rate parameter κ determines the relative volatilities of long and short rate. A high value of κ causes short-term rate movements to damp out quickly, so long-term volatility is reduced. If σ is assumed to be a function of time, it is also possible to fit the current term structure of volatility.

Under arbitrage-free conditions, the value of an interest rate contingent claim $V(r,t)$ satisfies the following PDE

$$\begin{cases} V_t + \frac{1}{2}\sigma^2 V_{rr} + (\theta(t) - \kappa r(t))V_r - rV = 0, \\ V(r,T) = \Phi(r). \end{cases} \tag{1.2}$$

For the bond value we have $\Phi(r) = B$ where B is the bond face value. A European-style option has a payoff $\Phi(r) = [\phi(Z(S,T)B - K)]^+$ at option expiry time S, $S < T$, where binary unit $\phi = +1$ for call and $\phi = -1$ for put, and K is a strike, $Z(S,T)$ is a zero coupon bond with maturity date T.

The Hull-White model is appealing and popular because it has analytical solution for vanilla option values.

In the Hull-White model the bond value is given by

$$Z(t,T) = e^{A(t,T) - B(t,T)r} \tag{1.3}$$

where

$$\begin{aligned} B(t,T) &= \frac{1}{\kappa}\left(1 - e^{-\kappa(T-t)}\right), \\ A(t,T) &= \ln\left[\frac{Z(0,T)}{Z(0,t)}\right] + B(t,T)f(0,t) - \frac{\sigma^2 B^2(t,T)(1 - e^{-2\kappa t})}{4\kappa}, \end{aligned}$$

and $f(0,t)$ is forward rate observed on the market.

The t-value a European-style option is given by

$$V(t,S,T,K,B) = \phi\left[Z(t,T)BN(d) - Z(t,S)KN(\phi(d - \delta))\right] \tag{1.4}$$

where

$$\delta = \frac{1}{\kappa}\left(1 - e^{-\kappa(T-S)}\right)\sqrt{\frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa(S-t)}\right)};\ d = \frac{1}{\delta}\ln\left[\frac{Z(t,T)B}{Z(t,S)K}\right] + \frac{1}{2}\delta.$$

and $N(x)$ is the normal cumulative density function.

2 Finite-Difference Method

We use the standard Crank-Nicolson discretization. To approximate the PDE (1.2), we construct a grid of size $M \times N$, $V(i,k)$, where $k = 0, ..., M$ is the time direction and $i = 0, ..., N$ is the space direction. Both bond and option prices are computed on the same grid. The bond price at time $t = T$ is subject to the following condition $V(i,0) = B,\ i = 0, ..., N$, where B is the bond face value. The option price at time $t = S$ is subject to the following condition

$$V(i,0) = \max\{\phi(B(i,t) - K), 0\} \tag{2.1}$$

where binary unit $\phi = +1$ for call and $\phi = -1$ for put.

This mean that if the pricing model does not have closed-form formulas for the bond price, we, first of all, need to compute bond price starting from $t = T$, where T is the bond maturity, up to $t = S$, where S is the option expiry. Then we have to apply initial condition for call price. The latter means to replace the bond values with option values computed by (2.1) on the same grid. Then we compute option prices using the grid up to $t = 0$.

If we want to price American option, we need to construct the full grid containing the bond prices, B, and the separate grid with option prices, V. Then we have to apply the initial condition (2.1) at time $t = S$ and apply the following condition at every time and space step

$$V(i,k) = \max\{\phi(B(i,k) - K), V(i,k)\} \tag{2.2}$$

We use the following boundary conditions

$$\frac{\partial^2 V}{\partial r^2} = 0 \tag{2.3}$$

This gives that $V_0^{k+1} = 2V_1^{k+1} - V_0^{k+2}$ and $V_N^{k+1} = 2V_{N-1}^{k+1} - V_0^{N-2}$.

The condition states that for small and large values of r, the value of contingent claim changes linearly with the r. The advantage of using these conditions is that they are universal ones and apply for both bond and option values.

3 The Hull-White Tree

The Hull-White tree enjoys popularity among market practitioners. A detailed analysis of the tree can be found in the Hull (2000).

The Hull-White tree is a general algorithm for the discrete-time implementation of diffusion models of the form

$$dx(t) = (\theta(t) - \kappa(t)x)dt + \sigma(t)dW. \tag{3.1}$$

If $x = r$, we get the Hull-White spot rate model.

The aim is to develop a discrete-time version that has the following properties.

1. It has a recombining trinomial tree structure. This allows changing the direction of the tree in order to prevent negative interest rates.

2. It converges to the continuous-time model (3.1).

3. It replicates a given initial term structure of interest rates of the type

$$Z^*(0,T) = \mathbb{E}\left[e^{-\int_0^T x(t)dt}\right]. \tag{3.2}$$

4 The Hull-White Model via Finite-Differences

We will play special attention to implementation of the Crank-Nicolson method with time-dependent coefficients for pricing options within the Hull-White model of the spot rate. Then we will compare results with those obtained by using the Hull-White trees. The reason for our interest is that FD method is a more general approach while Hull-White trees can be implemented only for a certain class of interest rate models and are subject to some stability criterion.

To implement the Hull-White model via FD method, we have to determine analytically the time-varying long-term mean. The solution is given by [Hull (2000)]

$$\theta(t) = \frac{\partial f(0,t)}{\partial t} + \kappa f(0,t) + \frac{\sigma^2}{2\kappa}(1 - e^{-\kappa t}). \tag{4.1}$$

The forward rate can be approximated accordingly to its definition as

$$f(0,t_k) \approx \frac{-\ln Z(0,t_k + \Delta t) - \ln Z(0,\Delta t)}{\Delta t}. \tag{4.2}$$

The derivative of the instantaneous forward rate w.r.t t can be approximated by

$$\left.\frac{\partial f(0,t)}{\partial t}\right|_{t=t_k} \approx \frac{f(0,t_k + \Delta t) - f(0,t_k - \Delta t)}{2\Delta t}. \tag{4.3}$$

Since, as a rule, we use discrete term structure, the numerical approximation of partial derivatives may cause instability of θ. If the yield curve is obtained by linear interpolation, then $f(0,t)$, $f_t(t,0)$ and thus $\theta(t)$ are not well defined at the nodes of the linear interpolation. As a result, the computed values of $\theta(t)$ will have oscillations near the nodes of the interpolation. This may cause serious problems for the accuracy of the numerical solutions, as Figure 1 shows. We see, that using piece-wise linear interpolation leads to oscillations of $\theta(t)$. However, spline also leads to some oscillation of $\theta(t)$ for small t. The reason for this may be the surplus of data points for short maturities.

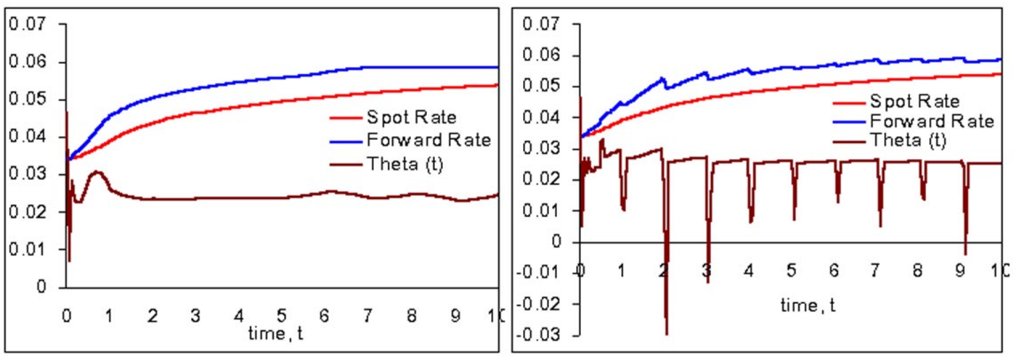

Figure 1: Calculation of theta using 1) Cubic spline, 2) Linear interpolation.

5 Numerical Results

In our analysis we use estimated yield curve given in Appendix A and parameters $\kappa = 0.35$ and $\sigma = 0.35$. The mean reversion parameter and volatility are rather higher than usual ones observed on the markets, where $\kappa \approx 0.01 - 0.1$, and $\sigma \approx 0.01 - 0.05$. Taking into account that the smaller are parameters the better should be numerical results, actual results may turn out to be better. We use piecewise linear interpolation in the HW tree and cubic spline in the Crank-Nicolson FD method with $r_{min} = -0.5$, $r_{max} = 0.5$ and the number of time and space steps set equal to the number of tree steps in the HW tree.

Table 1 reports pricing absolute errors (analytic price - model price), relative errors (absolute error / analytic price) for European call prices and prices of American put on bond obtained by using Hull-White model with tree and FD methods. The option strike is 70 CU and expiry is 5 years. The bond maturity is 10 years and the face value is 100 CU. Figure 2 shows the corresponding numbers.

NoSteps	HW errors	%	FD errors	%	HW price	FD price
50	0.0814	1.5062%	-0.0926	-1.7124%	12.9613	11.9712
100	0.0188	0.3477%	-0.0421	-0.7795%	12.0547	12.0093
200	0.0181	0.3346%	-0.0190	-0.3508%	11.9372	11.9741
400	0.0073	0.1342%	-0.0082	-0.1517%	11.9388	11.9569
500	0.0021	0.0392%	-0.0061	-0.1134%	11.9439	11.9535
800	0.0054	0.0997%	-0.0031	-0.0568%	11.9412	11.9484
1000	0.0023	0.0429%	-0.0021	-0.0383%	11.9453	11.9467
1600	0.0026	0.0477%	-0.0006	-0.0106%	11.9443	11.9441
2000	0.0007	0.0136%	-0.0001	-0.0015%	11.9468	11.9433

Table 1: Pricing errors of vanilla call and prices of American put.

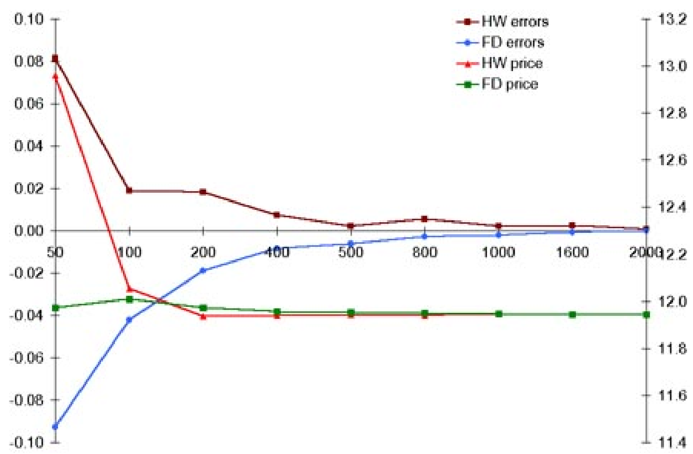

Figure 2: On the left axis are shown pricing errors of vanilla call. On the right axis are shown prices of American put.

We see that the FD method converges faster than the HW tree. We also see that as the number of nodes goes up, the accuracy of the FD approximation becomes independent of the number of nodes at a much faster rate than for the HW tree. As a rule, this is the case for other values of the term structure and option parameters. We also note that the FD method simultaneously gives us the option price and some hedge ratios: delta (the derivative wrt r), gamma (the second derivative wrt r), and theta (the derivative wrt t).

6 Conclusions

In our project we implemented the finite-difference method and the Hull-White tree algorithm. We found that for Vasicek and CIR models with constant coefficient the Crank-Nicolson method converges very quickly to analytical solutions. We showed that for Hull-White spot rate model with the time-dependent long-term mean the Crank-Nicolson method is, as a rule, superior to the Hull-White tree.

References

[1] Hull J (2000), "Options, Futures, and Other Derivative Securities," *5th. edition. New Jersey: Prentice-Hall.*

A Yield Curve and Stripping Data

In Table 2 we give our results of yield curve stripping using market quotes for EURIBOR (Euro Area Interbank Market Interest Rates) of 02 April, 2002.

Money rates	r_m	Date	T	$Z(t,T)$	$r(t,T)$
t/n	3.3450%	04-04-02	0.0056	0.99981	3.4406%
1m	3.3600%	06-05-02	0.0944	0.99683	3.4039%
2m	3.4000%	04-06-02	0.1750	0.99408	3.4374%
3m	3.4500%	04-07-02	0.2583	0.99117	3.4818%
6m	3.5930%	04-10-02	0.5139	0.98188	3.6082%
Swap rates	r_s				
1y	3.9975%	04-04-03	1.0194	0.96138	3.9171%
2y	4.4700%	05-04-04	2.0389	0.91579	4.3745%
3y	4.7375%	04-04-05	3.0500	0.86967	4.6419%
4y	4.9175%	04-04-06	4.0639	0.82420	4.8237%
5y	5.0525%	04-04-07	5.0778	0.77997	4.9618%
6y	5.1625%	04-04-08	6.0944	0.73713	5.0739%
7y	5.2575%	06-04-09	7.1139	0.69553	5.1748%
8y	5.3325%	05-04-10	8.1250	0.65629	5.2555%
9y	5.3950%	04-04-11	9.1361	0.61896	5.3237%
10y	5.4425%	04-04-12	10.1528	0.58383	5.3741%
11y	5.4850%	04-04-13	11.1667	0.55039	5.4216%

Table 2: The Yield Curve Data and Stripping Results.