Cohort Effect Structure in the Lee-Carter Residual Term

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ABSTRACT

The Lee-Carter Method is one of the most popular methodologies for forecasting mortality rate. However, if a cohort effect exists, the forecasted results will become less accurate, since this method smooths out such effect. This paper identifies the cohort effect structure in the Lee-Carter residual term and extrapolates it in such forecasting.

KEYWORDS

Lee-Carter Method; Residual Term; Cohort Effect; Time Series Model; AR(*p*) Model; Yule-Walker Equations; Extrapolation

1. INTRODUCTION

The Lee-Carter Method is one of the most popular methodologies for forecasting mortality rates. The method is widely known to be simple and has been used very successfully on U.S. data. However, on U.K. data, it has been found that a cohort effect weakens the goodness of fit, since the method smooths out such an effect. If the effect is significant, the Lee-Carter Method might not be an appropriate forecasting model.

The purpose of this paper is to identify the cohort effect structure in the Lee-Carter residual term and to extrapolate it in such forecasting. We used Japanese mortality data, for which it was shown by Willet (2004) et al. that cohort effects exist, to demonstrate the value of our approach. In section 2, we recognize how the cohort effect appears graphically in the residual term. Then we show that the Lee-Carter Method drives the cohort effect out to the residual term. In section 3, we identify the cohort effect structure by using a time series methodology. We will find that the cohort structure can be expressed as an AR(1) model. In section 4, we extrapolate the identified cohort structure in the subsequent forecasting. We will find that extrapolating the cohort structure in Lee-Carter forecasting will make the forecast become more natural. In conclusion, we emphasize this useful feature of the Lee-Carter residual term.

2. LEE-CARTER METHOD AND ITS RESIDUAL TERM

2.1 The Lee-Carter Methodology

The Lee-Carter methodology for forecasting mortality rates is a simple bilinear model in the variables x(age) and t (calendar year) of the following form:

$$ln(m_{x,t}) = a_x + b_x k_t + \varepsilon_{x,t}$$

where $m_{x,t}$ is the observed central death rate at age x in year t,

a_x describes the average shape of the age profile,

b_x describes the pattern of deviations from the age profile as the k_t varies,

k_t describes the change in overall mortality, and

$\varepsilon_{x,t}$ is the residual term at age x and time t.

In order to get a unique solution, usually the following constraints are applied:

$$\Sigma\, b_x = 1 \text{ and } \Sigma\, k_t = 0.$$

Given these constraints, the model can be fit by minimizing $\Sigma\, \varepsilon_{x,t}{}^2$. The simplest way to minimize the sum is to set a_x equal to the averages of the $ln(m_{x,t})$, then to get b_x and k_t from the first term of a singular value decomposition of the matrix:

$$\boldsymbol{R} = (\, ln(m_{x,t}) - a_x\,)\ .$$

In forecasting mortality rate, we need to model k_t as a stochastic time series process. It is known that k_t is well-modeled as a random walk with drift:

$$k_t = c + k_{t-1} + u_t.$$

The data used in this paper comes from the Abridged Life Tables for Japan 1962-2004 (Ministry of Health, Labor and Welfare). The goodness of fit is examined by the coefficient of determination (R^2). R^2 measures the proportion of the variation in $ln(m_{x,t})$ which is explained by the estimated a_x, b_x and k_t. The estimated values of a_x, b_x, k_t , and R^2 are shown in **Figure 1** to **Figure 4**.

Figure 1
Estimated Values of a_x Coefficients

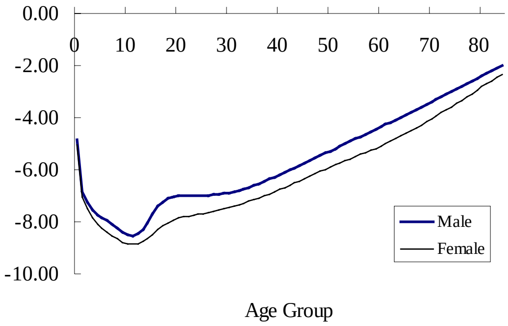


Figure 2
Estimated Values of b_x Coefficients

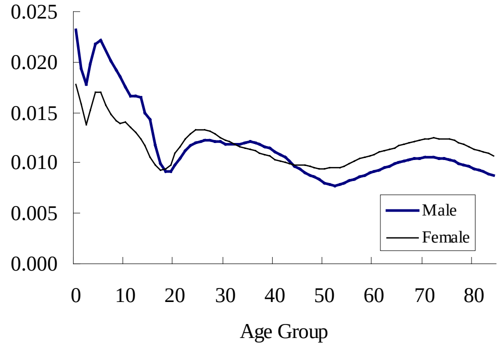


Figure 3
Estimated Values of k_t Coefficients

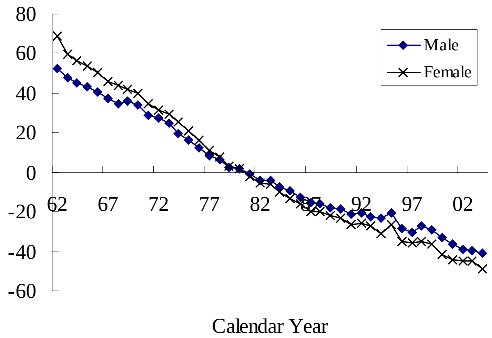


Figure 4
Coefficients of Determination

2.2 Graphical Representation of Lee-Carter Residual Term

Figure 5 and **Figure 6** are the graphical representations of the residual term. In order to highlight the relevant features, the term is standardized. In general, the cohort effect is defined to be a factor unique to year of birth. One can easily recognize a cohort effect as a diagonal line in these Figures. Thus we can assume that Lee-Carter Method may drive any cohort effect out to its residual term. Notice that the diagonal lines for the male residual term appear more explicit than those of the female residual term. This fact indicates that the male cohort effect may be stronger than female cohort effect in this data set. We will discuss again in section 3 the relative strength of these cohort effects.

Figure 5
Lee-Carter Residual Term - Male

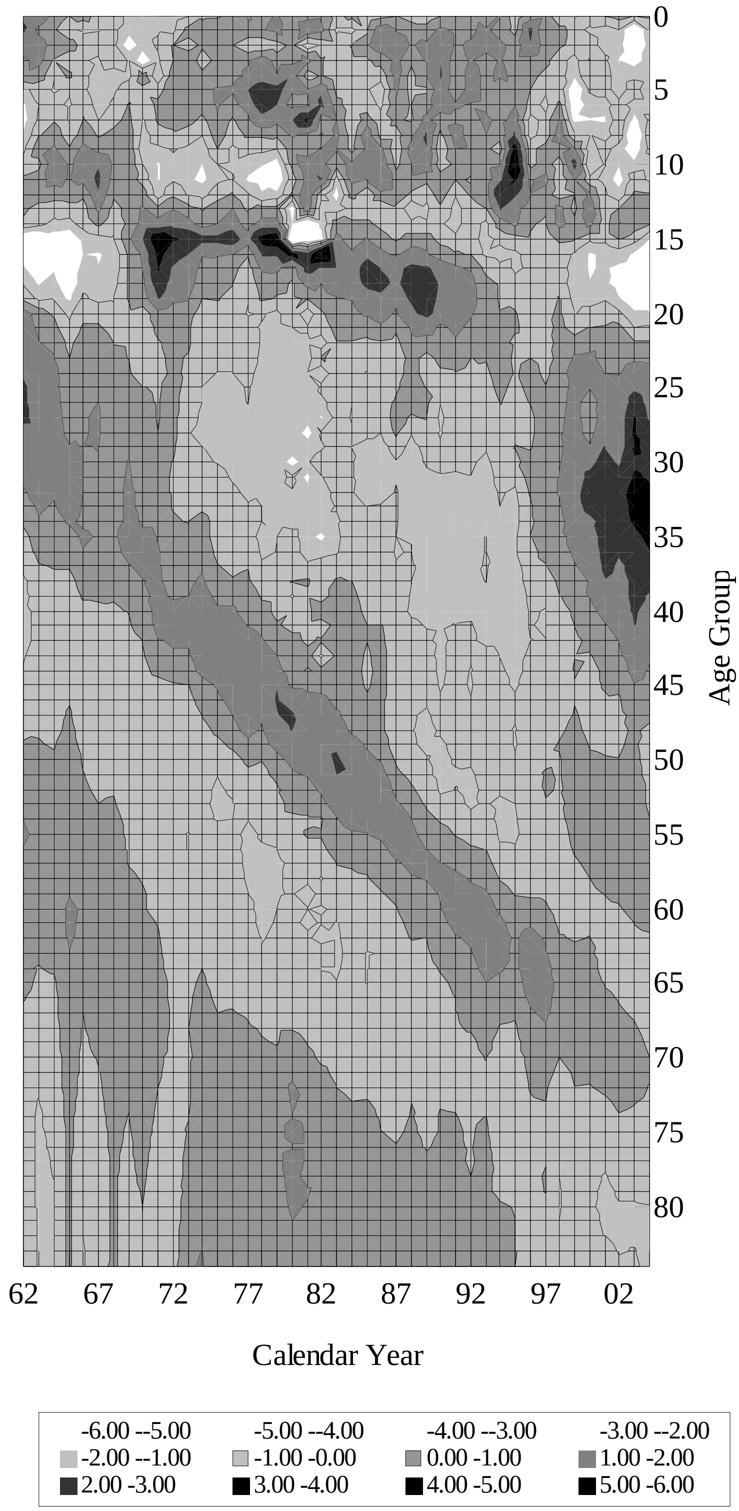
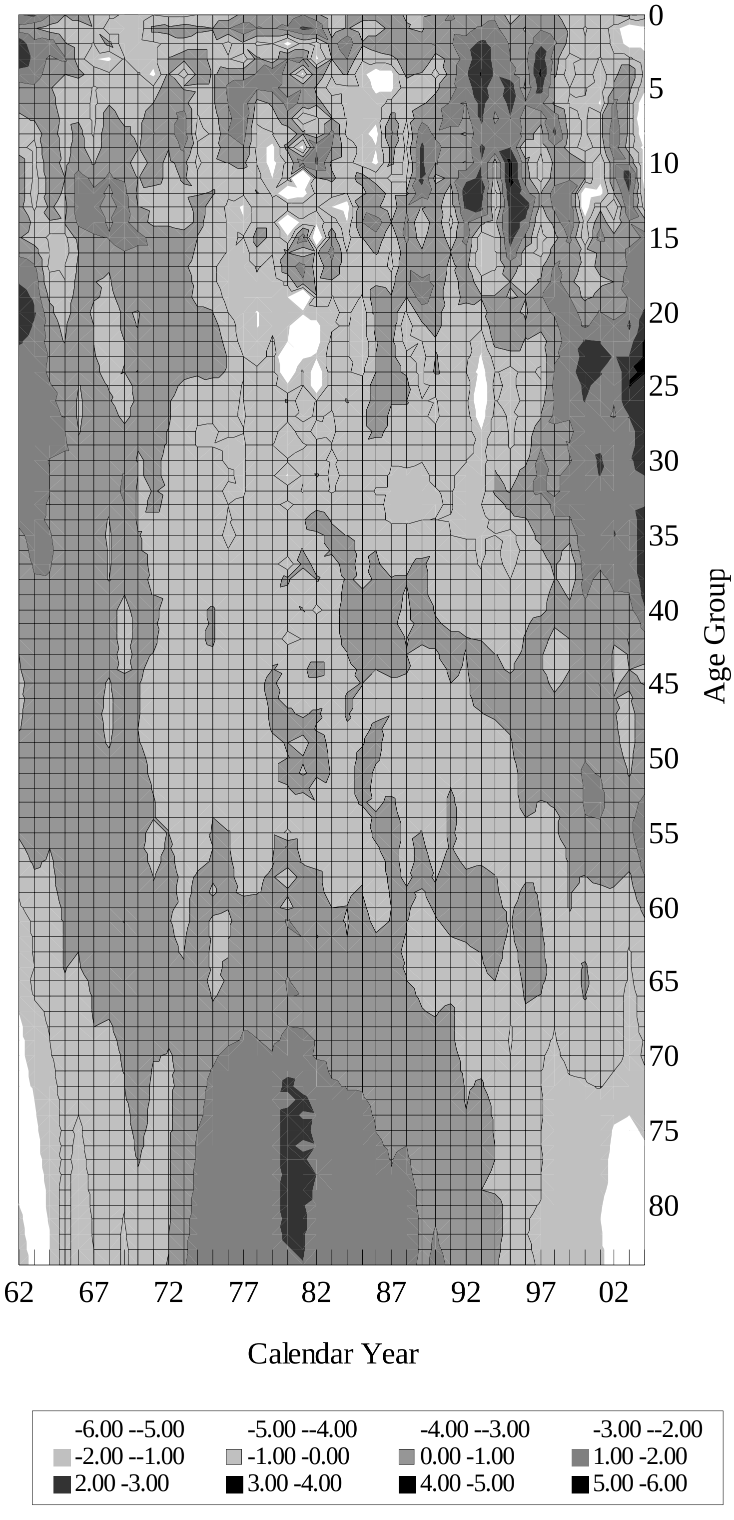


Figure 6
Lee Carter Residual Term - Female



2.3 The Lee-Carter Method Drives Any Cohort Effect out to its Residual Term

This is proved by the next lemma, since the residual term is the remainder after the matrix $\boldsymbol{R} = (\, ln(\, m_{x,t}\,) - a_x\,)$ is decomposed to $(b_x k_t)$. The lemma says that any (non-zero) factor which is unique to year of birth cannot be decomposed to the form of $(b_x k_t)$. Thus we conclude that a cohort effect, if any, must be concentrated in the residual term under Lee-Carter Method.

Lemma. *Let* $\boldsymbol{R} = (b_x k_t)$ *be an* $n \times T$ $(\, n \geq T\,)$ *matrix with the following conditions*;

(1) $b_x \geq 0$ *for any* x

(2) $\sum b_x = 1$

(3) $\sum k_t = 0$

(4) $b_i\, k_j = b_{i+1}\, k_{j+1}$ *for any* i, j

then $\boldsymbol{R}$ *becomes the zero matrix.*

Proof. $k_1 = 1 \times k_1 = (\sum b_x) \times k_1 = (\, b_1\, k_1 + b_2\, k_1 + b_3\, k_1 + \cdots + b_T\, k_1\,) + (\, b_{T+1}\, k_1 + \cdots + b_n\, k_1\,)$

$$= (\, b_T\, k_T + b_T\, k_{T-1} + b_T\, k_{T-2} + \cdots + b_T\, k_1\,) + (\, b_{T+1}\, k_1 + \cdots + b_n\, k_1\,) \quad (\text{ by } (4)\,)$$

$$= b_T \sum k_t + (\, b_{T+1}\, k_1 + \cdots + b_n\, k_1\,) = (\, b_{T+1} + \cdots + b_n\,)\, k_1 \quad (\text{ by } (3)\,)$$

Similarly we can obtain the following:

$$k_2 = (\, b_{T+1} + \cdots + b_n\,)\, k_2\,,\ \cdots\,,\ k_T = (\, b_{T+1} + \cdots + b_n\,)\, k_T$$

i) if $b_{T+1} + \cdots + b_n \neq 1$, then $k_t = 0$ for any t. Thus $\boldsymbol{R}$ becomes the zero matrix.

ii) if $b_{T+1} + \cdots + b_n = 1$, then $b_1 + \cdots + b_T = 0$ by (2). Thus $b_1 = \cdots = b_T = 0$ by (1).

As of the T th and (T+1) th rows of $\boldsymbol{R}$, the sum of each row becomes zero by (3) :

$$0 = \qquad b_T k_1 + \cdots + b_T k_{T-1} + b_T k_T$$

$$0 = b_{T+1} k_1 + b_{T+1} k_2 + \cdots + b_{T+1} k_T$$

Since $b_T k_1 = b_{T+1} k_2$,..., $b_T k_{T-1} = b_{T+1} k_T$ by (4), $b_{T+1} k_1$ is equal to $b_T k_T$.

Thus the (T+1) th row can be written ;

$$b_T k_T, b_T k_1, \cdots, b_T k_{T-1}$$

These are zero because $b_T = 0$, i.e. the (T+1)th row of ***R*** becomes zero.

Inductively, any more than the T th row becomes zero as well.

Hence ***R*** becomes the zero matrix. ■

Assuming b_x to be non-negative is the key. However, this condition is reasonable, because any negative b_x would mean that the mortality rate for a certain x always shows an adverse trend against the aggregate trend k_t, which is unlikely to occur in the real world over an extended period of time. The analysis of the Japanese mortality data have shown that all coefficients b_x are well in excess of zero (see Figure 2).

3. COHORT EFFECT STRUCTURE

3.1 Time Series Methodology

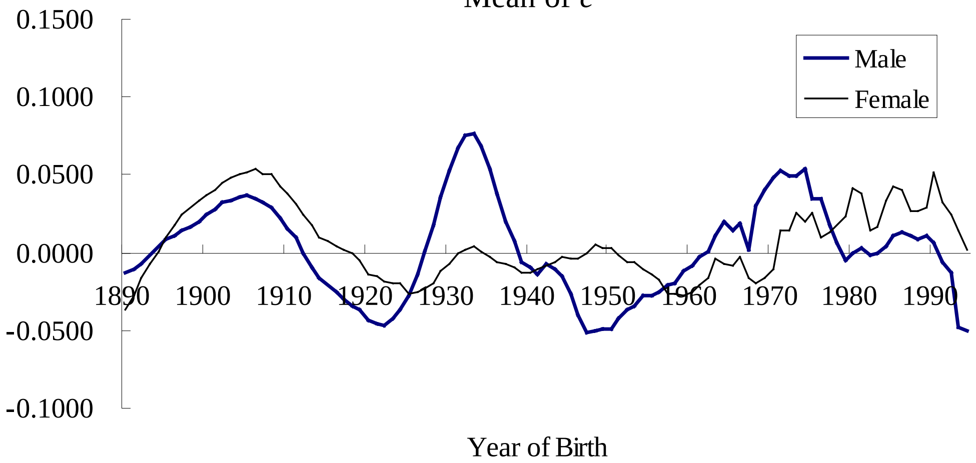
In order to identify the cohort effect structure in the residual term, we start by sorting the term by year of birth. Let T be year of birth and denote $\varepsilon_{x,T+x}$ as $c^T(x)$:

$$c^T(x) = \varepsilon_{x,T+x}.$$

The sequence $c^T(x)$, $c^T(x+1)$, $c^T(x+2)$,… is well defined and unique to T. This is considered to be a time series process of x with mean μ_T. We show μ_T in **Figure 7.** There we find that the μ_T for male are more volatile than those for female.

Figure 7

Mean of c^T



In order to identify a time series process, the autocorrelation function is often examined. The *k*-th autocorrelation function ρ_k is defined as follows:

$$\rho_k = \gamma_k / \gamma_0,$$

$$\gamma_k = E((c^T(x) - \mu_T)(c^T(x-k) - \mu_T)).$$

Figure 8 and **Figure 9** are the exhibits of ρ_k (k = 0, 1, 2, 3,…,10). Roughly speaking, they begin at ρ_0 =1 and then decline geometrically. This is a typical feature of an AR(p) time series process (see Appendix). Thus we select the AR(p) model for $c^T(x)$:

$$c^T(x) = \phi_1 c^T(x-1) + \phi_2 c^T(x-2) + \ldots + \phi_p c^T(x-p) + \delta + \varepsilon^T(x),$$

$$\mu_T = \delta / (1 - \phi_1 - \phi_2 - \ldots - \phi_p).$$

Once the model is selected, then we need to know the degree p. In order to determine the degree, the *Yule-Walker Equations* are applied:

Yule-Walker Equations

$$\rho_1 = \phi_1 + \phi_2 \rho_1 + \ldots + \phi_p \rho_{p-1}$$

....

$$\rho_p = \phi_1 \rho_{p-1} + \phi_2 \rho_{p-2} + \ldots + \phi_p$$

For indices k greater than p we have,

$$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2} + ... + \phi_p \rho_{k-p}$$

We begin by hypothesizing that p=1. Then the equation boils down to $\rho_1 = \phi_1$. Thus, if the calculated value ϕ_1 is significantly different from zero, we know that the autoregressive process is at least order 1. Next consider the hypothesis that p=2. A new set of ϕ_1 and ϕ_2 estimates are given by solving the Yule-Walker Equations for p=2. Now, if ϕ_2 is significantly different from zero, we know that the autoregressive process is at least order 2. We repeat this process for successive values of p. To test whether a particular ϕ_j is zero, we can use the fact that it is approximately normally distributed, with mean zero and variance 1/(the number in the sample). We can check whether it is statistically significant at the 5 percent level by determining whether it exceeds $2/\sqrt{\text{(the number in the sample)}}$ in magnitude.

The results are shown in **Table 1** and **Table 2**. From these tables, although there are a few exceptions, the degree p generally becomes 1. In addition, we find that the coefficients ϕ_1 for male are higher than those for female if T is greater than 1930. Now we go back to Figure 5 and Figure 6. From the shape of these figures, we can say the cohort effect is strong for a certain T if the following two conditions are satisfied at the same time:

1) the absolute value of its mean (μ_T) is enough large and
2) the coefficient ϕ_1 for c^T is close to 1.

Figure 8

Autocorrelation Function of c^T for Male

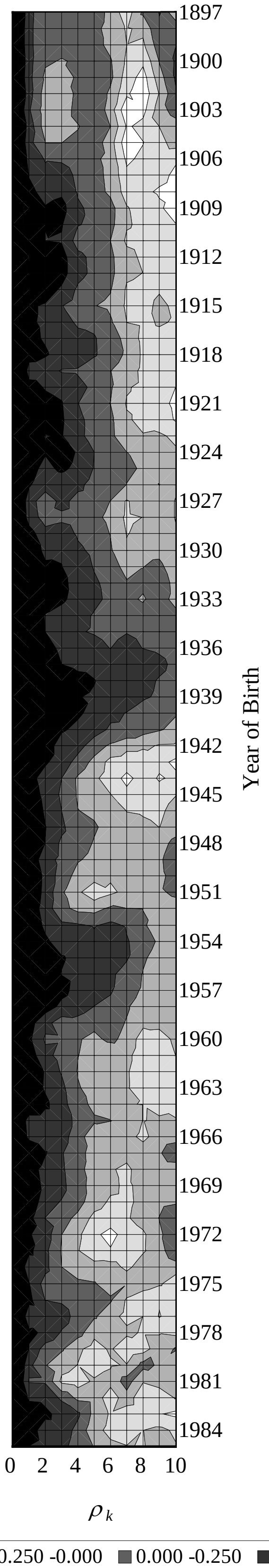


Figure 9

Autocorrelation Function of c^T for Female

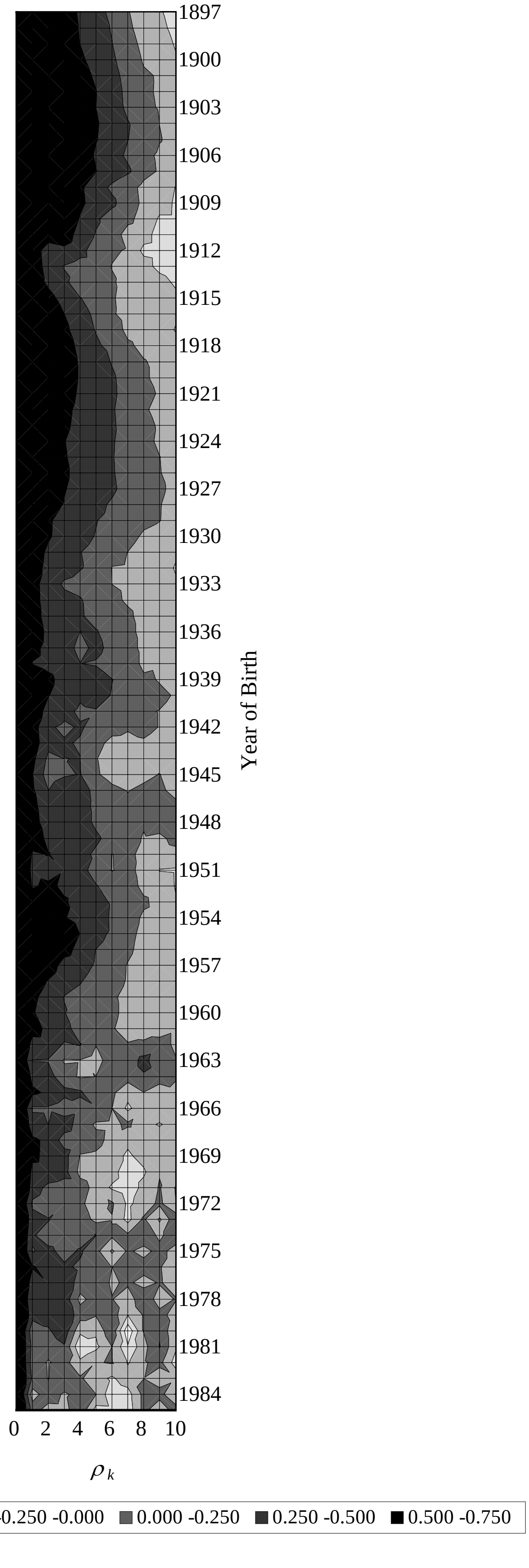


Table 1
Autocorrelation Function and Estimates of ϕ_k for Male

Year of Birth	Autocorrelation Function						Degree			
T	ρ_0	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	p	ϕ_1	ϕ_2	ϕ_3
1887	1.000	-0.035	-0.084	0.180	-0.238	-0.107	0	-	-	-
1888	1.000	0.022	-0.042	0.115	-0.158	-0.092	0	-	-	-
1889	1.000	0.013	-0.190	0.177	-0.020	0.245	0	-	-	-
1890	1.000	0.322	-0.194	-0.068	0.102	0.150	0	-	-	-
1891	1.000	0.336	-0.187	-0.093	0.075	0.171	0	-	-	-
1892	1.000	0.289	-0.111	-0.026	0.022	0.104	0	-	-	-
1893	1.000	0.258	-0.135	0.198	0.145	-0.027	0	-	-	-
1894	1.000	0.299	-0.075	0.137	0.346	0.096	0	-	-	-
1895	1.000	0.395	0.003	0.152	0.238	0.242	0	-	-	-
1896	1.000	0.367	0.017	0.170	0.192	0.178	0	-	-	-
1897	1.000	0.327	0.052	0.222	0.122	0.144	0	-	-	-
1898	1.000	0.315	0.064	0.094	0.041	0.067	0	-	-	-
1899	1.000	0.311	0.070	0.061	0.019	0.065	0	-	-	-
1900	1.000	0.330	0.028	-0.013	0.023	0.099	0	-	-	-
1901	1.000	0.319	-0.046	-0.121	0.039	0.222	0	-	-	-
1902	1.000	0.356	-0.112	-0.208	0.088	0.219	1	0.356	-	-
1903	1.000	0.282	-0.126	-0.175	0.113	0.209	0	-	-	-
1904	1.000	0.390	-0.115	-0.182	-0.021	0.195	1	0.390	-	-
1905	1.000	0.344	-0.005	-0.004	0.090	0.146	1	0.344	-	-
1906	1.000	0.431	0.226	0.204	0.067	0.055	1	0.431	-	-
1907	1.000	0.487	0.324	0.413	0.176	0.097	1	0.487	-	-
1908	1.000	0.573	0.402	0.450	0.224	0.102	1	0.573	-	-
1909	1.000	0.617	0.520	0.592	0.286	0.180	1	0.617	-	-
1910	1.000	0.595	0.503	0.523	0.263	0.205	1	0.595	-	-
1911	1.000	0.633	0.495	0.476	0.147	0.065	1	0.633	-	-
1912	1.000	0.718	0.650	0.593	0.317	0.183	1	0.718	-	-
1913	1.000	0.716	0.667	0.596	0.326	0.193	1	0.716	-	-
1914	1.000	0.612	0.603	0.463	0.208	0.084	2	0.388	0.366	-
1915	1.000	0.525	0.479	0.258	0.126	0.007	1	0.525	-	-
1916	1.000	0.519	0.478	0.334	0.217	0.187	1	0.519	-	-
1917	1.000	0.544	0.484	0.420	0.312	0.271	1	0.544	-	-
1918	1.000	0.572	0.518	0.441	0.419	0.278	1	0.572	-	-
1919	1.000	0.413	0.407	0.226	0.217	0.080	0	-	-	-
1920	1.000	0.574	0.483	0.438	0.329	0.190	1	0.574	-	-
1921	1.000	0.613	0.565	0.513	0.255	0.046	1	0.613	-	-
1922	1.000	0.655	0.572	0.531	0.339	0.107	1	0.655	-	-
1923	1.000	0.624	0.491	0.512	0.325	0.125	1	0.624	-	-
1924	1.000	0.640	0.515	0.590	0.480	0.248	1	0.640	-	-
1925	1.000	0.585	0.442	0.540	0.451	0.236	1	0.585	-	-
1926	1.000	0.528	0.360	0.370	0.314	0.150	1	0.528	-	-
1927	1.000	0.358	0.104	0.321	0.166	0.121	1	0.358	-	-
1928	1.000	0.389	0.149	0.203	0.168	0.061	1	0.389	-	-
1929	1.000	0.612	0.331	0.348	0.221	0.136	1	0.612	-	-
1930	1.000	0.634	0.453	0.460	0.293	0.199	1	0.634	-	-
1931	1.000	0.659	0.530	0.507	0.372	0.234	1	0.659	-	-
1932	1.000	0.698	0.606	0.570	0.399	0.284	1	0.698	-	-
1933	1.000	0.750	0.603	0.543	0.409	0.300	1	0.750	-	-
1934	1.000	0.676	0.489	0.414	0.318	0.230	1	0.676	-	-
1935	1.000	0.692	0.499	0.376	0.272	0.228	1	0.692	-	-
1936	1.000	0.711	0.573	0.460	0.378	0.343	1	0.711	-	-
1937	1.000	0.724	0.616	0.499	0.459	0.441	1	0.724	-	-
1938	1.000	0.793	0.709	0.610	0.556	0.508	1	0.793	-	-
1939	1.000	0.754	0.681	0.570	0.506	0.479	1	0.754	-	-
1940	1.000	0.773	0.717	0.585	0.531	0.436	1	0.773	-	-

Autocorrelation Function and Estimates of ϕ_k for Male - Continued

Year of Birth	Autocorrelation Function						Degree			
T	ρ_0	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	p	ϕ_1	ϕ_2	ϕ_3
1941	1.000	0.751	0.688	0.526	0.453	0.352	1	0.751	-	-
1942	1.000	0.710	0.624	0.396	0.344	0.127	1	0.710	-	-
1943	1.000	0.626	0.532	0.387	0.159	-0.075	1	0.626	-	-
1944	1.000	0.594	0.394	0.249	-0.039	-0.206	1	0.594	-	-
1945	1.000	0.655	0.430	0.213	-0.027	-0.128	1	0.655	-	-
1946	1.000	0.706	0.469	0.248	0.005	-0.032	1	0.706	-	-
1947	1.000	0.713	0.505	0.304	0.043	0.022	1	0.713	-	-
1948	1.000	0.648	0.496	0.198	0.090	-0.014	1	0.648	-	-
1949	1.000	0.695	0.360	0.178	0.019	-0.059	1	0.695	-	-
1950	1.000	0.708	0.440	0.125	-0.086	-0.156	1	0.708	-	-
1951	1.000	0.723	0.434	0.039	-0.207	-0.396	1	0.723	-	-
1952	1.000	0.664	0.396	0.063	-0.065	-0.089	1	0.664	-	-
1953	1.000	0.694	0.443	0.292	0.272	0.233	1	0.694	-	-
1954	1.000	0.693	0.544	0.390	0.408	0.309	1	0.693	-	-
1955	1.000	0.771	0.575	0.514	0.459	0.360	1	0.771	-	-
1956	1.000	0.771	0.593	0.494	0.446	0.342	1	0.771	-	-
1957	1.000	0.831	0.684	0.565	0.422	0.344	1	0.831	-	-
1958	1.000	0.744	0.606	0.449	0.375	0.246	1	0.744	-	-
1959	1.000	0.596	0.335	0.119	0.162	0.067	1	0.596	-	-
1960	1.000	0.551	0.213	0.222	0.021	-0.060	1	0.551	-	-
1961	1.000	0.624	0.332	0.159	0.005	-0.238	1	0.624	-	-
1962	1.000	0.722	0.473	0.227	-0.014	-0.137	1	0.722	-	-
1963	1.000	0.651	0.467	0.313	0.020	-0.101	1	0.651	-	-
1964	1.000	0.800	0.556	0.340	0.099	-0.068	1	0.800	-	-
1965	1.000	0.355	0.354	0.350	0.196	0.040	0	-	-	-
1966	1.000	0.423	0.279	0.356	0.119	-0.189	1	0.423	-	-
1967	1.000	0.791	0.518	0.271	0.078	-0.093	1	0.791	-	-
1968	1.000	0.630	0.411	0.293	0.045	-0.073	1	0.630	-	-
1969	1.000	0.634	0.443	0.316	0.081	-0.083	1	0.634	-	-
1970	1.000	0.687	0.363	0.233	0.094	-0.141	1	0.687	-	-
1971	1.000	0.589	0.245	0.110	-0.067	-0.320	1	0.589	-	-
1972	1.000	0.518	0.397	0.003	-0.197	-0.451	1	0.518	-	-
1973	1.000	0.563	0.316	-0.027	-0.236	-0.425	1	0.563	-	-
1974	1.000	0.307	0.321	0.000	-0.123	-0.145	0	-	-	-
1975	1.000	0.508	0.154	0.028	0.043	0.012	1	0.508	-	-
1976	1.000	0.570	0.243	0.219	0.095	0.126	1	0.570	-	-
1977	1.000	0.345	0.353	0.361	0.150	0.037	0	-	-	-
1978	1.000	0.551	0.456	0.231	0.109	-0.192	1	0.551	-	-
1979	1.000	0.477	0.140	0.034	-0.203	-0.341	1	0.477	-	-
1980	1.000	0.319	0.015	-0.159	-0.257	-0.403	0	-	-	-
1981	1.000	0.232	0.227	-0.308	-0.370	-0.209	0	-	-	-
1982	1.000	0.435	0.403	0.206	-0.064	-0.105	0	-	-	-
1983	1.000	0.728	0.575	0.392	0.293	-0.075	1	0.728	-	-
1984	1.000	0.657	0.354	0.374	0.179	-0.167	1	0.657	-	-
1985	1.000	0.495	0.403	0.346	0.110	-0.025	0	-	-	-
1986	1.000	0.343	0.372	0.148	-0.087	0.371	0	-	-	-
1987	1.000	0.348	0.161	0.073	-0.207	0.340	0	-	-	-
1988	1.000	0.002	-0.021	0.175	-0.316	0.061	0	-	-	-
1989	1.000	0.044	-0.260	-0.146	0.315	-0.307	0	-	-	-
1990	1.000	0.397	-0.113	-0.135	0.097	0.127	1	0.397	-	-
1991	1.000	0.329	0.159	0.166	-0.366	-0.005	0	-	-	-
1992	1.000	0.505	0.477	0.355	-0.199	-0.381	0	-	-	-
1993	1.000	0.698	0.437	0.110	-0.155	-0.571	1	0.698	-	-
1994	1.000	0.347	0.422	-0.204	0.049	-0.521	0	-	-	-
1995	1.000	0.118	0.092	-0.229	0.105	-0.139	0	-	-	-

Table 2

Autocorrelation Function and Estimates of ϕ_k for Female

Year of Birth	Autocorrelation Function						Degree			
T	ρ_0	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	p	ϕ_1	ϕ_2	ϕ_3
1887	1.000	0.487	0.276	0.151	-0.189	-0.464	0	-	-	-
1888	1.000	0.480	0.360	0.077	0.101	-0.241	0	-	-	-
1889	1.000	0.530	0.299	0.187	0.035	0.036	0	-	-	-
1890	1.000	0.584	0.316	0.138	0.143	-0.041	1	0.584	-	-
1891	1.000	0.716	0.410	0.192	0.119	0.034	1	0.716	-	-
1892	1.000	0.738	0.551	0.286	0.148	0.008	1	0.738	-	-
1893	1.000	0.763	0.587	0.411	0.218	0.030	1	0.763	-	-
1894	1.000	0.795	0.641	0.465	0.329	0.094	1	0.795	-	-
1895	1.000	0.805	0.675	0.524	0.384	0.209	1	0.805	-	-
1896	1.000	0.816	0.693	0.545	0.447	0.272	1	0.816	-	-
1897	1.000	0.823	0.706	0.559	0.483	0.344	1	0.823	-	-
1898	1.000	0.823	0.716	0.550	0.493	0.365	1	0.823	-	-
1899	1.000	0.842	0.737	0.577	0.499	0.387	1	0.842	-	-
1900	1.000	0.858	0.745	0.615	0.540	0.413	1	0.858	-	-
1901	1.000	0.874	0.765	0.634	0.577	0.469	1	0.874	-	-
1902	1.000	0.893	0.766	0.647	0.592	0.506	1	0.893	-	-
1903	1.000	0.888	0.769	0.648	0.593	0.500	1	0.888	-	-
1904	1.000	0.880	0.804	0.695	0.630	0.522	1	0.880	-	-
1905	1.000	0.849	0.797	0.697	0.643	0.511	1	0.849	-	-
1906	1.000	0.848	0.797	0.697	0.624	0.473	1	0.848	-	-
1907	1.000	0.858	0.848	0.769	0.671	0.506	2	0.492	0.426	-
1908	1.000	0.809	0.759	0.680	0.543	0.365	1	0.809	-	-
1909	1.000	0.816	0.776	0.747	0.554	0.380	1	0.816	-	-
1910	1.000	0.759	0.694	0.689	0.480	0.282	1	0.759	-	-
1911	1.000	0.673	0.604	0.645	0.392	0.234	1	0.673	-	-
1912	1.000	0.626	0.391	0.440	0.335	0.111	1	0.626	-	-
1913	1.000	0.661	0.397	0.236	0.154	0.137	1	0.661	-	-
1914	1.000	0.641	0.450	0.304	0.114	0.108	1	0.641	-	-
1915	1.000	0.805	0.576	0.418	0.259	0.145	1	0.805	-	-
1916	1.000	0.797	0.674	0.501	0.330	0.204	1	0.797	-	-
1917	1.000	0.813	0.673	0.567	0.357	0.237	1	0.813	-	-
1918	1.000	0.814	0.690	0.593	0.453	0.283	1	0.814	-	-
1919	1.000	0.848	0.697	0.596	0.480	0.381	1	0.848	-	-
1920	1.000	0.847	0.747	0.611	0.484	0.383	1	0.847	-	-
1921	1.000	0.850	0.715	0.605	0.467	0.381	1	0.850	-	-
1922	1.000	0.825	0.681	0.557	0.444	0.349	1	0.825	-	-
1923	1.000	0.834	0.657	0.544	0.436	0.385	1	0.834	-	-
1924	1.000	0.785	0.632	0.509	0.412	0.389	1	0.785	-	-
1925	1.000	0.805	0.646	0.519	0.421	0.346	1	0.805	-	-
1926	1.000	0.813	0.666	0.538	0.428	0.378	1	0.813	-	-
1927	1.000	0.810	0.652	0.516	0.420	0.366	1	0.810	-	-
1928	1.000	0.779	0.599	0.487	0.369	0.308	1	0.779	-	-
1929	1.000	0.754	0.534	0.407	0.280	0.256	1	0.754	-	-
1930	1.000	0.719	0.526	0.407	0.291	0.242	1	0.719	-	-
1931	1.000	0.653	0.455	0.322	0.253	0.214	1	0.653	-	-
1932	1.000	0.599	0.434	0.337	0.260	0.199	1	0.599	-	-
1933	1.000	0.605	0.360	0.218	0.177	0.089	1	0.605	-	-
1934	1.000	0.613	0.378	0.307	0.268	0.156	1	0.613	-	-
1935	1.000	0.624	0.402	0.406	0.271	0.187	1	0.624	-	-
1936	1.000	0.655	0.442	0.379	0.248	0.264	1	0.655	-	-
1937	1.000	0.577	0.430	0.343	0.191	0.303	1	0.577	-	-
1938	1.000	0.490	0.424	0.360	0.252	0.235	1	0.490	-	-
1939	1.000	0.668	0.561	0.406	0.293	0.339	1	0.668	-	-
1940	1.000	0.673	0.511	0.386	0.278	0.325	1	0.673	-	-

Autocorrelation Function and Estimates of ϕ_k for Female - Continued

Year of Birth	Autocorrelation Function						Degree			
T	ρ_0	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	p	ϕ_1	ϕ_2	ϕ_3
1941	1.000	0.654	0.419	0.304	0.218	0.237	1	0.654	-	-
1942	1.000	0.626	0.276	0.217	0.267	0.199	1	0.626	-	-
1943	1.000	0.634	0.322	0.265	0.237	0.068	1	0.634	-	-
1944	1.000	0.565	0.197	0.248	0.254	0.013	1	0.565	-	-
1945	1.000	0.510	0.128	0.222	0.259	0.026	1	0.510	-	-
1946	1.000	0.535	0.250	0.421	0.368	0.172	1	0.535	-	-
1947	1.000	0.562	0.367	0.448	0.351	0.193	1	0.562	-	-
1948	1.000	0.572	0.406	0.343	0.381	0.201	1	0.572	-	-
1949	1.000	0.654	0.439	0.427	0.400	0.296	1	0.654	-	-
1950	1.000	0.458	0.510	0.363	0.331	0.208	0	-	-	-
1951	1.000	0.420	0.352	0.479	0.377	0.096	1	0.420	-	-
1952	1.000	0.460	0.569	0.437	0.391	0.267	0	-	-	-
1953	1.000	0.699	0.632	0.536	0.360	0.338	1	0.699	-	-
1954	1.000	0.668	0.608	0.511	0.439	0.443	1	0.668	-	-
1955	1.000	0.657	0.630	0.632	0.494	0.343	2	0.427	0.350	-
1956	1.000	0.649	0.629	0.589	0.421	0.245	2	0.416	0.359	-
1957	1.000	0.531	0.632	0.413	0.357	0.222	0	-	-	-
1958	1.000	0.649	0.480	0.341	0.268	0.147	1	0.649	-	-
1959	1.000	0.593	0.347	0.242	0.174	0.031	1	0.593	-	-
1960	1.000	0.537	0.292	0.267	0.079	-0.002	1	0.537	-	-
1961	1.000	0.597	0.437	0.353	0.118	0.082	1	0.597	-	-
1962	1.000	0.395	0.405	0.223	0.262	0.008	0	-	-	-
1963	1.000	0.239	0.233	-0.026	-0.018	-0.110	0	-	-	-
1964	1.000	0.421	0.340	0.111	-0.030	0.004	1	0.421	-	-
1965	1.000	0.551	0.458	0.311	0.278	0.101	1	0.551	-	-
1966	1.000	0.224	0.219	0.134	0.188	0.238	0	-	-	-
1967	1.000	0.332	0.247	0.361	0.155	-0.040	0	-	-	-
1968	1.000	0.547	0.444	0.151	0.113	0.080	1	0.547	-	-
1969	1.000	0.543	0.444	0.337	-0.015	-0.037	1	0.543	-	-
1970	1.000	0.447	0.436	0.346	-0.077	-0.179	0	-	-	-
1971	1.000	0.389	0.178	0.127	0.127	-0.099	1	0.389	-	-
1972	1.000	0.205	0.098	0.219	0.065	-0.162	0	-	-	-
1973	1.000	0.328	0.263	0.072	0.166	-0.074	0	-	-	-
1974	1.000	0.279	0.111	0.078	0.103	0.257	0	-	-	-
1975	1.000	0.236	0.317	0.158	0.281	0.082	0	-	-	-
1976	1.000	0.515	0.428	0.291	0.210	0.050	1	0.515	-	-
1977	1.000	0.373	0.493	0.426	0.143	0.219	0	-	-	-
1978	1.000	0.287	0.362	0.400	-0.068	0.104	0	-	-	-
1979	1.000	0.343	0.368	0.444	0.118	0.020	0	-	-	-
1980	1.000	0.092	0.215	0.402	-0.186	-0.206	0	-	-	-
1981	1.000	0.160	0.099	0.225	-0.471	-0.313	0	-	-	-
1982	1.000	0.118	-0.022	0.090	-0.196	-0.011	0	-	-	-
1983	1.000	0.213	-0.014	0.225	0.035	-0.151	0	-	-	-
1984	1.000	-0.113	0.155	-0.040	0.083	0.000	0	-	-	-
1985	1.000	0.123	-0.121	-0.095	0.203	-0.351	0	-	-	-
1986	1.000	-0.122	-0.219	0.072	-0.266	-0.113	0	-	-	-
1987	1.000	-0.075	-0.087	-0.082	-0.051	-0.279	0	-	-	-
1988	1.000	0.308	0.257	0.249	-0.324	-0.322	0	-	-	-
1989	1.000	0.457	0.005	-0.109	-0.042	0.165	1	0.457	-	-
1990	1.000	0.362	0.265	0.004	-0.264	-0.032	0	-	-	-
1991	1.000	0.206	-0.173	-0.067	0.246	0.303	0	-	-	-
1992	1.000	-0.081	-0.175	-0.107	-0.056	-0.060	0	-	-	-
1993	1.000	0.075	-0.304	-0.128	-0.098	0.349	0	-	-	-
1994	1.000	0.123	-0.451	-0.096	0.156	0.529	0	-	-	-
1995	1.000	-0.010	-0.386	-0.152	-0.166	0.672	0	-	-	-

4. EXTRAPOLATION OF COHORT STRUCTURE

4.1 Forecasting Methodology

Once the coefficient ϕ is given in the AR(1) model, we can forecast the future $c^T(x)$ by

$$c^T(x+1) = \mu_T + \phi \times (c^T(x) - \mu_T).$$

Since the absolute value of estimated ϕ is less than 1, the process is stationary. The stationary AR(1) process is equivalent to a moving average process of infinite order and thus with infinite memory, which is consistent with the nature of a cohort effect.

4.2 Extrapolation of Cohort Structure in Lee-Carter Forecasting

Figure 10 and Figure 11 show the extrapolated cohort structures on a raw Lee-Carter forecast. Since the extrapolation of cohort structure is independent with respect to k_t, one can assume any kind of future aggregate mortality trend simply by formulating k_t. From the shape of these figures, we find that extrapolating the cohort structure in Lee-Carter forecasting will make the forecast become more natural.

Figure 10
Extrapolated Cohort Effect Structure - Male

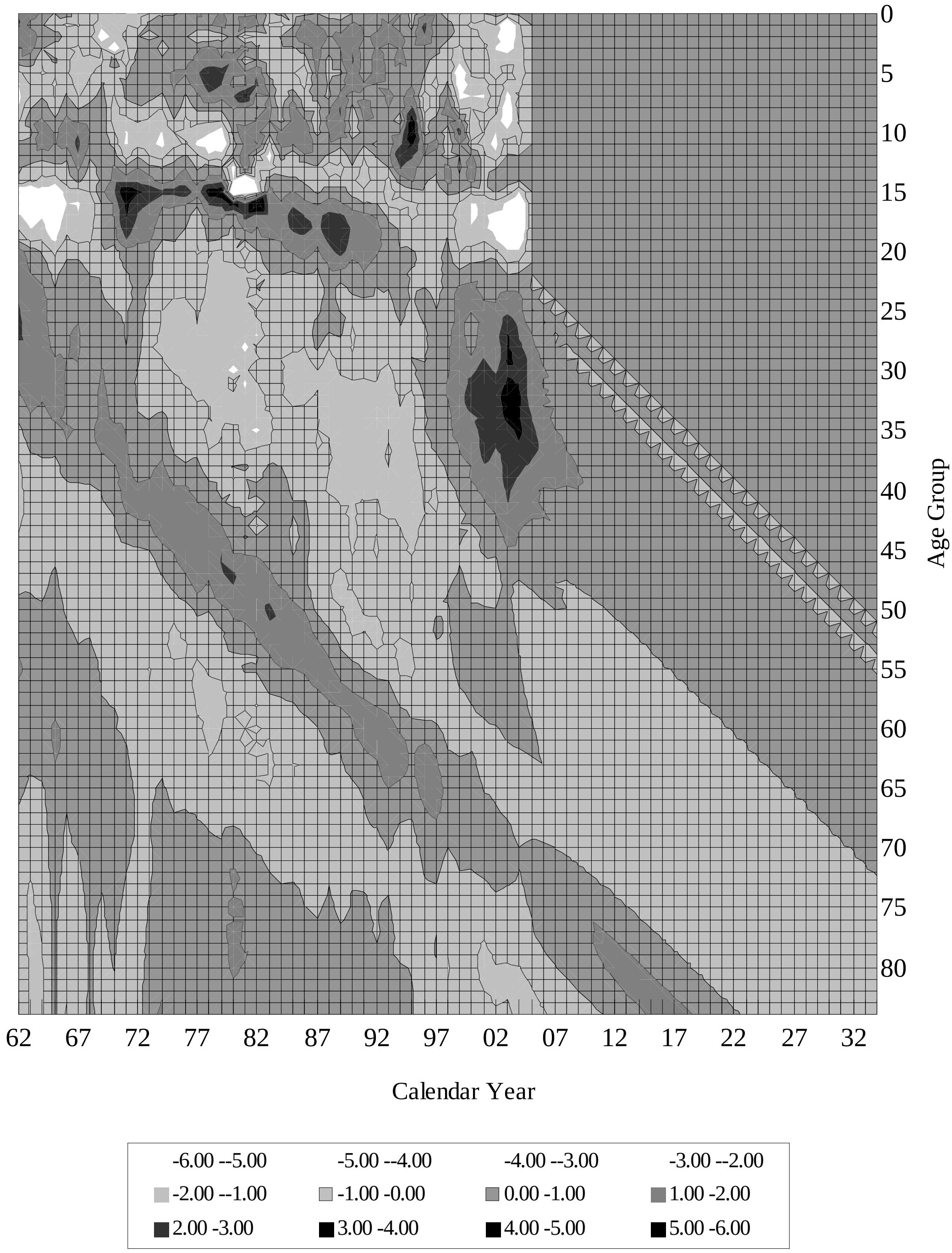
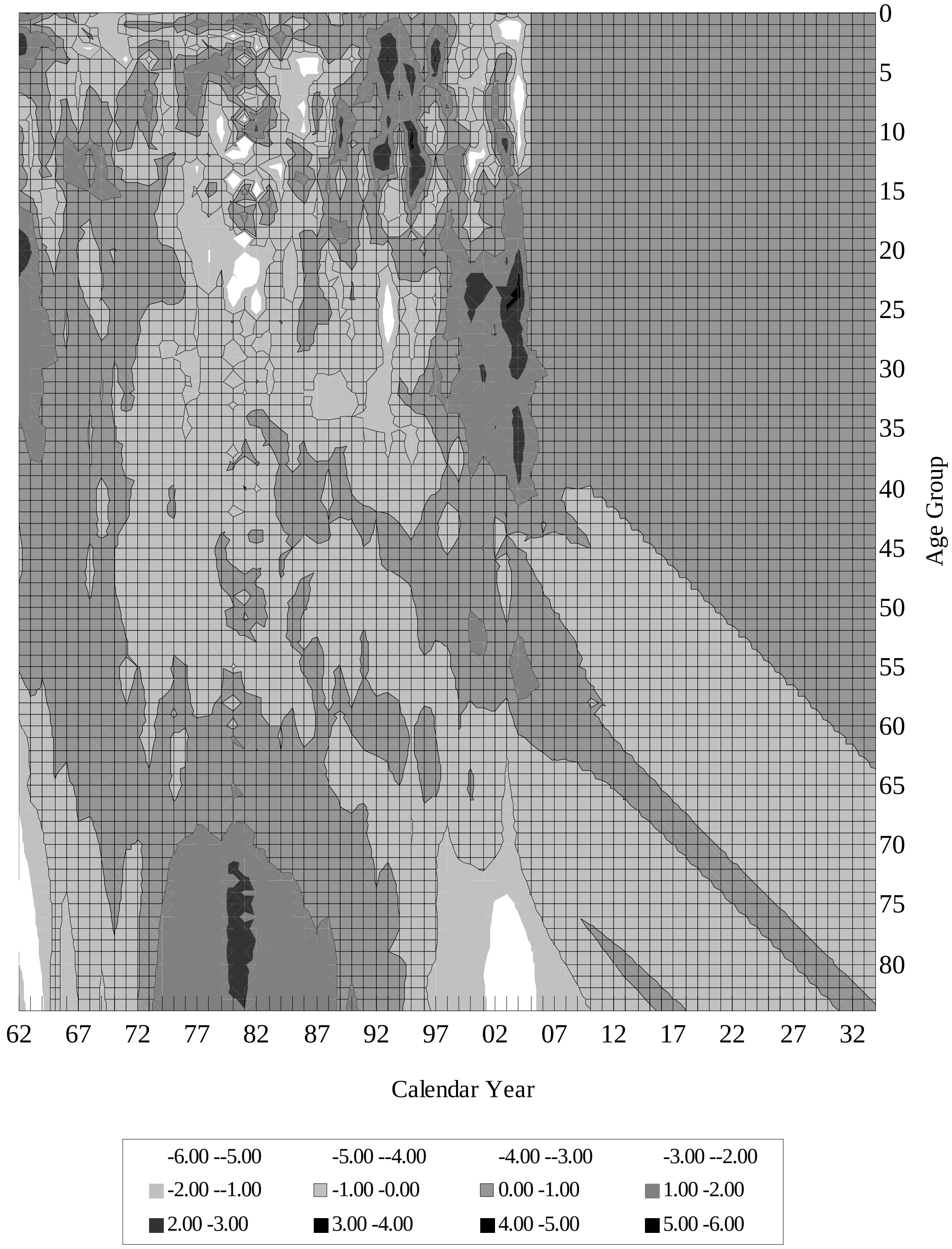


Figure 11
Extrapolated Cohort Effect Structure - Female



5. CONCLUSION

The Lee-Carter Method is one of the most useful methodologies for forecasting mortality rate. However, there has been a difficulty in applying the Method if a cohort effect exists. Here, by identifying the cohort effect concentrated in its residual term, we extrapolate such structure in forecasting. Thus, we can conclude that the Lee-Carter Method can be used successfully even in data in which a cohort effect exists.

The feature that the Lee-Carter residual term drives the cohort effect out to its residual term is very interesting. This feature will be useful not only to analyze the cohort effect, but also to make the mortality forecasting more accurate.

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APPENDIX

THE AUTOCORRELATION FUNCTIONS OF TIME SERIES MODELS

1) MA(q) Model : $y_t = \mu + \varepsilon_t - \theta_1\varepsilon_{t-1} - \theta_2\varepsilon_{t-2} - \ldots - \theta_q\varepsilon_{t-q}$

$\rho_k = (-\theta_k + \theta_1\theta_{k+1} + \ldots + \theta_{q-k}\theta_q) / (1+\theta_1^2 + \theta_2^2 + \ldots + \theta_q^2)$ for $k = 1,\ldots,q$

$= 0$ for $k > q$

For the example q=1, $\rho_1 = (-\theta_1) / (1+\theta_1^2)$; $\rho_2 = \rho_3 = \ldots = 0$

2) AR(p) Model : $y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \ldots + \phi_p y_{t-p} + \delta + \varepsilon_t$

$\rho_1 = \phi_1 + \phi_2\rho_1 + \ldots + \phi_p\rho_{p-1}$

....

$\rho_p = \phi_1 \rho_{p-1} + \phi_2\rho_{p-2} + \ldots + \phi_p$

$\rho_k = \phi_1 \rho_{k-1} + \phi_2\rho_{k-2} + \ldots + \phi_p\rho_{k-p}$ for $k > p$

For the example p=1, $\rho_1 = \phi_1$, $\rho_2 = \phi_1^2$, ... , $\rho_k = \phi_1^k$

3) ARMA(p,q) Model :

Since q is the memory of the moving average part of the process, for $k \geq q+1$ the autocorrelation function exhibits the properties of a purely autoregressive process.

For the example p=q=1,

$\rho_1 = (1-\phi_1\theta_1)(\phi_1 - \theta_1)/(1+\theta_1^2 - 2\phi_1\theta_1)$, $\rho_k = \phi_1 \rho_{k-1}$ for $k \geq 2$