Risk analysis and hedging in incomplete markets

DISSERTATION

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ABSTRACT

Variable annuities are in the spotlight in today's insurance market. The tax deferral feature and the absence of the investment risk for the insurer (while keeping the possibility of investment benefits) boosted their popularity. They represent the sensible way found by the insurance industry to compete with other stock market and financial intermediaries. A variable annuity is an investment wrapped with a life insurance contract. An insurer who sells variable annuities bears two different types of risk. On one hand, he deals with a financial risk on the investment. On the other hand there exists an actuarial (mortality) risk, given by the lifetime of the insured. Should the insured die, the insurer has to pay a possible claim, depending on the options elected (return of premium, reset, ratchet, roll-up). In the Black-Scholes model, the share price is a continuous function of time. Some rare events (which are rather frequent lately), can accompany jumps in the share price. In this case the market model is incomplete and hence there is no perfect hedging of options. I considered a simple market model with one riskless asset and one risky asset, whose price jumps in different proportions at some random times which correspond to the jump times of a Poisson process. Between the jumps the risky asset follows the Black-Scholes model. The mathematical model consists of a probability space, a Brownian motion and a Poisson process. The jumps are independent and identically distributed. The approach consists of defining a notion of risk and choosing a price and a hedge in

order to minimize the risk. In the dual market (insurance and financial) the risk-minimizing strategies defined by Follmer and Sondermann and the work of Moller with equity-linked insurance products are reviewed and used for variable annuities, with death or living benefits.

The theory of incomplete markets is complex and intriguing. There are many interesting connections between such models and game theory, while the newest and maybe the most powerful research tool comes from economics, the utility function (tastes and preferences).

This is dedicated to my family.

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CHAPTER 1

INTRODUCTION

1.1 Introduction and General Settings

The approach throughout this thesis is based on the concept of arbitrage. It is a remarkably simple concept and it is independent of preferences of the **actors** in the financial market.

The basic assumption is that everybody prefers more to less and that any increase in consumption opportunities must somehow be paid for.

The core background for our exposition is the risk-neutral (probabilistic) pricing of derivatives securities. A derivative (or contingent claim) is a financial contract whose value at expiration date T (or expiry) is determined by the price of an underlying financial asset at time T. In this chapter we discuss the basics for pricing contingent claims. The general assumption of this chapter is that we are in the classic Black-Scholes model, which means that, according to the fundamental theorem of asset pricing, the price of any contingent claim can be calculated as the discounted expectation of the corresponding payoff with respect to the equivalent martingale measure.

1.2 Discrete time financial mathematics

In this section we will consider a discrete-time model.

We consider a finite probability space $(\Omega, \mathcal{F}, P)$, with $|\Omega|$ a finite number, and for any $\omega \in \Omega \quad P(\{\omega\}) > 0$. We have a time horizon T, which is the terminal date for all economic activities considered. We use a filtration F of σ - algebras $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_T$ and we take $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_T = \mathcal{F} = \mathcal{P}(\Omega)$ the power set of Ω. This financial market contains $d+1$ financial assets. One is a risk-free asset (a bond or a bank account for example) labeled 0 and d are risky assets (stocks) labeled 1 to d. The prices of these assets at time t: $S_0(t,\omega), S_1(t,\omega), \ldots, S_d(t,\omega)$ are non-negative and $\mathcal{F}_t$ - measurable. Let $S(t) = (S_0(t), \ldots, S_d(t))$ denote the vector of prices at time t. We assume $S_0(t)$ is strictly positive for all $t \in \{0, 1, \ldots, T\}$ and also assume that $S_0(0) = 1$. We define $\beta(t) = \frac{1}{S_0(t)}$ as a discount factor.

We have then constructed a market model $\mathcal{M}$ consisting of a probability space $(\Omega, \mathcal{F}, P)$, a set of trading dates, a price process S, and the information structure F.

Definition 1.2.1 A *trading strategy* (or *dynamic portfolio*) φ is a R^{d+1} vector stochastic process $\varphi = \big(\varphi_0(t,\omega), \varphi_1(t,\omega), \ldots, \varphi_d(t,\omega)\big)_{t=1}^{T}$ which is predictable: each $\varphi_i(t)$ is $\mathcal{F}_{t-1}$ - measurable for $t \geq 1$, where $\varphi_i(t)$ denotes the number of shares of asset i held in the portfolio at time t and which is to be determined on the basis of information available before time t (predictability).

Definition 1.2.2 The value of the portfolio at time t is the scalar product

$$V_\varphi(t) = \varphi(t) \cdot S(t) = \sum_{i=0}^{d} \varphi_i(t) S_i(t) \qquad t = 1, 2, \ldots, T$$

and $\qquad V_\varphi(0) = \varphi(1) S(0)$

The process $V_\varphi(t,\omega)$ is called the *wealth* or *value process* of the trading strategy φ. We call $V_\varphi(0)$ the initial investment of the investor (endowment).

Definition 1.2.3 The *gains process* G_φ of a trading strategy φ is given by:

$$G_\varphi(t) = \sum_{x=1}^{t} \varphi(x)\Big[S(x) - S(x-1)\Big] \ , \qquad t = 1, 2, \ldots, T$$

If we define $\widetilde{S}(t) = \Big(1, \beta(t)S_1(t), \ldots, \beta(t)S_d(t)\Big)$ the vector of discounted prices we also have the discounted value process $\widetilde{V}_\varphi(t) = \varphi(t)\widetilde{S}(t)$ for $t = 1, 2, \ldots, T$ and we can see that the discounted gains process $\widetilde{G}_\varphi(t) = \sum_{x=1}^{t} \varphi(x)\Big[\widetilde{S}(x) - \widetilde{S}(x-1)\Big]$ reflects the gains from trading with assets 1 to d only.

Definition 1.2.4 The strategy φ is *self-financing*, $\varphi \in \Phi$, if

$$\varphi(t)S(t) = \varphi(t+1)S(t) \ , \qquad t = 1, 2, \ldots, T-1$$

This means that when new prices $S(t)$ are quoted at time t the investor adjusts his portfolio from $\varphi(t)$ to $\varphi(t+1)$, without bringing in or consuming any wealth.

To prove the fundamental theorem of asset pricing we need the following results, which are also interested and important by themselves:

Proposition 1.2.1 A trading strategy φ is self financing with respect to $S(t)$ if and only if φ is self-financing with respect to $\tilde{S}(t)$.

Proposition 1.2.2 A trading strategy φ is self financing if and only if

$$\tilde{V}(t) = V_\varphi(0) + \tilde{G}_\varphi(t)$$

The well-being of any market is given by the absence of arbitrage opportunities (arbitrage=free lunch).

Definition 1.2.5 Let $\Phi_0 \subset \Phi$ be a set of self-financing strategies. A strategy $\varphi \in \Phi_0$ is called an *arbitrage opportunity* or arbitrage strategy with respect to Φ_0 if

$P\Big\{V_\varphi(0) = 0\Big\} = 1$ and the terminal wealth satisfies

$$P\Big\{V_\varphi(T) \geq 0\Big\} = 1 \quad and \quad P\Big\{V_\varphi(T) > 0\Big\} > 0$$

We say that a security market $\mathcal{M}$ is *arbitrage-free* if there are no arbitrage opportunities in the class Φ.

Next we introduce the notion of "risk-neutral probability" which also also central in financial mathematics:

Definition 1.2.6 A probability measure P^* on $(\Omega, \mathcal{F}_T)$ equivalent to P is called a *martingale measure* for $\widetilde{S}$ if the process $\widetilde{S}$ follows a P^* - martingale with respect to the filtration F. We denote $\mathcal{P}\left(\widetilde{\mathcal{S}}\right)$ the class of equivalent martingale measures.

One proposition that follows quickly and is useful in proving Theorem 1.2.1 is:

Proposition 1.2.3 Let $P^* \in \mathcal{P}\left(\widetilde{\mathcal{S}}\right)$ and φ a self-financing strategy. Then the wealth process $\widetilde{V}(t)$ is a P^* martingale with respect to the filtration F.

The no-arbitrage theorem describes the necessary and sufficient conditions for no-arbitrage and makes the connection between the real world (finacial market) and theory (martingales):

Theorem 1.2.1 (No-Arbitrage Theorem) The market $\mathcal{M}$ is arbitrage-free if and only if there exists a probability measure P^* equivalent to P under which the discounted d - dimensional asset price process $\widetilde{S}$ is a P^* - martingale.

The question now is how we use this theorem to price contingent claims. We start with a definition:

Definition 1.2.7 A *contingent claim* X with maturity date T is an arbitrary non-negative $\mathcal{F}_T$ - measurable random variable.

We say that the claim is *attainable* if there exists a replicating strategy $\varphi \in \Phi$ such that

$$V_{\varphi}(T) = X$$

The following theorem is the first theoretical approach to pricing contingent claims:

Theorem 1.2.2 The arbitrage price process $\pi_X(t)$ of any attainable contingent claim X is given by the *risk-neutral valuation formula*:

$$\pi_X(t) = \beta(t)^{-1} E^* \Big(X \beta(T) \mid \mathcal{F}_t \Big) \quad for\ any \quad t = 0, 1, \ldots, T$$

where E^* is the expectation taken with respect to an equivalent martingale measure P^*.

Theorem 1.2.2 says that any attainable contingent claim can be priced using the equivalent martingale measure. So, clearly, “attainability” would be a very desirable property of any market. So the next definition follows naturally:

Definition 1.2.8 The market $\mathcal{M}$ is *complete* if every contingent claim is attainable.

The following theorem gives a nice characterization of a complete market:

Theorem 1.2.3 (Completeness Theorem) An arbitrage-free market $\mathcal{M}$ is complete if and only if there exists a unique probability measure P^* equivalent to P under which discounted asset prices are martingales.

Let’s summarize what we have seen so far.

Theorem 1.2.1 tells us that if the market is arbitrage-free, equivalent martingale measures P^* exist. Theorem 1.2.3 tells us that if the market is complete, equivalent martingale measures are unique. Putting them together we get:

Theorem 1.2.4 (Fundamental Theorem of Asset Pricing) In an arbitrary-free complete market $\mathcal{M}$, there exists a unique equivalent martingale measure P^*.

Finally, Theorem 1.2.2 gives us in the complet market setting the following:

Theorem 1.2.5 (Risk-Neutral Pricing Formula) In an arbitrage-free complete market $\mathcal{M}$, arbitrage prices of contingent claims are their discounted expected values under the risk-neutral (equivalent martingale) measure P^*.

1.3 Continuous time financial mathematics

We start with a general model of a frictionless securities market where investors are allowed to trade continuously up to some fixed finite planning horizon T. Uncertainty in this financial market is modeled by a probability space $(\Omega, \mathcal{F}, P)$ and a filtration F of σ - algebras $\mathcal{F}_t$, with $0 \leq t \leq T$, satisfying the usual conditions of right-continuity and completeness. We assume that $\mathcal{F}_0$ is trivial and that $\mathcal{F}_T = \mathcal{F}$.

There are d+1 primary traded assets, whose price processes are given by stochastic processes $S_0, \ldots, S_d$. We assume that $S = (S_0, \ldots, S_d)$ follows an adapted, right-continuous with left-limits (RCLL) and strictly positive semimartingale on $(\Omega, \mathcal{F}, P, F)$. We also assume that $S_0(t)$ is a non-dividend paying asset which is almost surely strictly positive and use it as a numeraire.

We denote by M(P) the financial market described above.

Definition 1.3.1 A *trading strategy* (or *dynamic portfolio*) φ is a R^{d+1} vector stochastic process $\varphi(t) = \big(\varphi_0(t), \varphi_1(t), \ldots, \varphi_d(t)\big)$, $0 \leq t \leq T$ which is predictable and locally bounded.

Here $\varphi_i(t)$ denotes the number of shares of asset i held in the portfolio at time t - determined on the basis of information available before time t. This means that the investor selects his time t portfolio after observing the prices $S(t-)$.

Definition 1.3.2 The *value* of the portfolio φ at time t is given by the scalar product

$$V_\varphi(t) = \varphi(t) \cdot S(t) = \sum_{i=0}^{d} \varphi(t) S_i(t), \qquad t \in [0, T]$$

Definition 1.3.3 The *gains process* $G_\varphi(t)$ is defined by

$$G_\varphi(t) = \int_0^t \varphi(u)dS(u) = \sum_{i=0}^{d} \int_0^t \varphi_i(u)dS_i(u)$$

Definition 1.3.4 A trading strategy is called *self-financing* if the value process satisfies

$$V_\varphi(t) = V_\varphi(0) + G_\varphi(t) \qquad for\ all \qquad t \in [0, T]$$

We can define the discounted price process, the discounted value process and discounted gains process with the help of the numeraire $S_0(t)$.

Definition 1.3.5 A self-financing strategy is called an *arbitrage opportunity* or *arbitrage strategy* if $V_\varphi(0) = 0$ and the terminal value satisfies

$$P\Big\{V_\varphi(T) \geq 0\Big\} = 1 \quad and \quad P\Big\{V_\varphi(T) > 0\Big\} > 0$$

Definition 1.3.6 We say that a probability measure Q defined on $(\Omega, \mathcal{F})$ is a (strong) equivalent martingale measure if Q is equivalent to P and the discounted process $\widetilde{S}$ is a Q-local martingale (martingale).

We denote the set of martingale measures by $\mathcal{P}$

Definition 1.3.7 A self financing strategy is called *tame* if $\widetilde{V}_\varphi(t) \geq 0$ for all $t \in [0, T]$. We denote by Φ the set of tame trading strategies.

The following proposition assures us that the existence of an equivalent martingale measure implies the absence of arbitrage.

Proposition 1.3.1 Assume $\mathcal{P}$ is not empty. Then the market model contains no arbitrage opportunities in Φ.

In order to get equivalence between the absence of arbitrage opportunities and the existence of an equivalent martingale measure we need some further definitions and requirements.

Definition 1.3.8 A *simple* predictable strategy is a predictable process which can be represented as a finite linear combination of stochastic processes of the form $\psi 1_{[\tau_1,\tau_2]}$ where τ_1 and τ_2 are stopping times and ψ is an $\mathcal{F}_{\tau_1}$-measurable random variable.

Definition 1.3.9 We say that a simple predictable trading strategy is δ-*admissible* if $V_\varphi(t) \geq -\delta$ for every $t \in [0,T]$.

Definition 1.3.10 A price process S satisfies NFLVR (no free lunch with vanishing risk) if for any sequence (φ_n) of simple trading strategies such that φ_n is δ_n-admissible and the sequence δ_n tends to zero, we have $V_{\varphi_n}(T) \to 0$ in probability as $n \to \infty$.

The following fundamental theorem of asset pricing is proved in [7]:

Theorem 1.3.1 (Fundamental Theorem of Asset Pricing - continuous time) There exists an equivalent martingale measure for the financial market model M(P) if and only if the condition NFLVR holds true.

For all the proofs of the above theorems refer to [3].

1.4 Options in the Black-Scholes model

Let us assume that we have a market with two assets, a riskless (B) and a risky one (S). The riskless asset (bond or savings account) is modelled by the following ordinary differential equation

$$dB_t = rB_t dt$$

where $r \geq 0$ is an instantaneous interest rate (different from the rate in the discrete models). Without loss of generality, we set $B_0 = 1$ and so $B_t = e^{rt}$ for $t \geq 0$. The risky asset is a stock (or stock index) whose price is modelled by the following stochastic differential equation:

$$dS_t = S_t(\mu dt + \sigma dZ_t),$$

where μ and σ are constants and Z_t is a standard Browninan motion. The model is described on a probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration F of σ - algebras $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_T$. We take $\mathcal{F}_t = \sigma\{S_u,\ u \leq t\}$, and so the price process for the stock is adapted to the filtration. We want to prove that there exists a probability measure $P^\star$ equivalent to P, under which the discounted share price $S_t^\star = e^{-rt}S_t$ is a martingale. We have:

$$dS_t^\star = -re^{-rt}S_t dt + e^{-rt}dS_t = S_t^\star((\mu - r)dt + \sigma dZ_t)$$

and if we set $W_t = Z_t + \frac{(\mu-r)t}{\sigma}$, we get

$$dS_t^\star = S_t^\star \sigma dW_t.$$

Now, recall the Girsanov theorem:

Theorem 1.4.1. Let $(\theta_t)_{0 \leq t \leq T}$ be an adapted process satisfying $\int_0^T \theta_s^2 ds < \infty$ a.s.

and such that the process $(L_t)_{0 \leq t \leq T}$ defined by

$$L_t = exp\Big(- \int_0^t \theta_s dZ_s - \frac{1}{2} \int_0^t \theta_s^2 ds \Big)$$

is a martingale. Then, under the probability P^L with density L_T relative to P, the process $(W_t)_{0 \leq t \leq T}$ defined by $W_t = Z_t + \int_0^t \theta_s ds$ is a standard Brownian motion.

For a proof of Girsanov's theorem, see [19].

Using this theorem with $\theta_t = \frac{\mu - r}{\sigma}$ we get that there exists a probability $P^\star$ under which $(W_t)_{0 \leq t \leq T}$ is a standard Brownian motion. Under this probability $P^\star$, the discounted price process $(S_t^\star)$ is a martingale and

$$S_t^\star = S_0^\star exp\Big(\sigma W_t - \frac{(\sigma)^2 t}{2} \Big).$$

Let us consider now a standard European call option. The option is defined by a non-negative, $\mathcal{F}_T$-measurable random variable $H = f(S_T) = (S_T - K)_+$, where K is the exercise price.

Theorem 1.4.2. In the Black-Scholes model, any option defined by a non-negative, $\mathcal{F}_T$-measurable random variable H, which is square-integrable under the probability $P^\star$, is replicable by a trading strategy and the value at time t of any replicating portfolio is given by:

$$V_t = E^\star \big(e^{-r(T-t)} H | \mathcal{F}_t \big).$$

Thus, the option value at time t can be naturally defined by the expression $E^\star \big(e^{-r(T-t)} H | \mathcal{F}_t \big)$.

Proof: We follow [21]. Let us assume that there is an admissible strategy (ϑ, η) replicating the option. The value of the portfolio at time t is given by

$$V_t = \vartheta_t B_t + \eta_t S_t,$$

and the terminal values are equal: $V_T = H$. Defining the discounted value $V_t^\star = V_t e^{-rt}$ we get

$$V_t^\star = \vartheta_t + \eta_t S_t^\star.$$

The strategy is self-financing and hence

$$V_t^\star = V_0 + \int_0^t \eta_u dS_u^\star = V_0 + \int_0^t \eta_u \sigma S_u^\star dW_u.$$

It can be shown that $(V_t^\star)$ is a square-integrable martingale under $P^\star$ and hence

$$V_t^\star = E^\star(V_T^\star | \mathcal{F}_t),$$

and so

$$V_t = E^\star(e^{-r(T-t)} H | \mathcal{F}_t).$$

So, if a portfolio (ϑ, η) replicates the option, its value is given by the above formula. Now it remain to show that the option is indeed replicable, i.e. there exist some processes (ϑ_t) and (η_t) such that

$$\vartheta_t B_t + \eta_t S_t = E^\star(e^{-r(T-t)} H | \mathcal{F}_t).$$

The process $M_t := E^\star(e^{-rt} H | \mathcal{F}_t)$ is a $P^\star$-square integrable martingale.

Now, using the representation theorem for martingales we obtain that there exists an adapted process $(K_t)_{0 \le t \le T}$ such that $E^\star(\int_0^T K_s^2 ds) < \infty$ and

$$M_t = M_0 + \int_0^t K_s dW_s \ a.s.$$

for any $t \in [0, T]$.

The strategy $\phi = (\vartheta, \eta)$ with $\eta_t = \frac{K_t}{(\sigma S_t^\star)}$ and $\vartheta_t = M_t - H_t S_t^\star$ is self-financing and its value at time t is given by

$$V_t(\phi) = e^{rt} M_t = E^\star(e^{-r(T-t)} H | \mathcal{F}_t).$$

In our case (European call), the random variable H can be written as $H = f(S_T) = (S_T - k)_+$ and we can express the option value V_t at time t as a function of t and S_t as follows:

$$V_t = E^\star(e^{-r(T-t)} f(S_T)|\mathcal{F}_t)$$
$$= E^\star\left(e^{-r(T-t)} f\left(S_t e^{r(T-t)} e^{\sigma(W_T - W_t) - (\frac{\sigma^2}{2})(T-t)}\right)\Big|\mathcal{F}_t\right).$$

The random variable S_t is $\mathcal{F}_t$-measurable and $W_T - W_t$ is independent of $\mathcal{F}_t$. A standard result in probability theory allows us to write

$$V_t = F(t, S_t),$$

where

$$F(t,x) = E^\star\left(e^{-r(T-t)} f\left(x e^{-r(T-t)} e^{\sigma(W_T - W_t) - (\frac{\sigma^2}{2})(T-t)}\right)\right).$$

As W_t is a standard Brownian motion, $W_T - W_t$ is a zero-mean normal variable with variance $T - t$ and so,

$$F(t,x) = e^{-r(T-t)} \int_{-\infty}^{\infty} f\left(x e^{(r - \frac{\sigma^2}{2})(T-t) + \sigma y \sqrt{T-t}}\right) \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dy.$$

Now F can be calculated explicitly for call options. In this case $f(x) = (x - K)_+$ and

$$F(t,x) = E^\star\left(e^{-r(T-t)}\left(e^{(r-\frac{\sigma^2}{2})(T-t) + \sigma(W_T - W_t)} - K\right)_+\right)$$
$$= E\left(x e^{\sigma\sqrt{\theta} g - \sigma^2 \frac{\theta}{2}} - K e^{-r\theta}\right)_+$$

where g is a standard Gaussian variable and $\theta = T - t$. We define:

$$d_1 = \frac{ln(x/K) + (r + \frac{\sigma^2}{2})\theta}{\sigma\sqrt{\theta}}$$

and

$$d_2 = d_1 - \sigma\sqrt{\theta}.$$

Then we have,

$$
\begin{aligned}
F(t,x) &= E\Big[\Big(xe^{\sigma\sqrt{\theta}g-\sigma^2\frac{\theta}{2}} - Ke^{-r\theta}\Big)I_{\{g+d_2\geq 0\}}\Big] \\
&= \int_{-d_2}^{\infty}\big(xe^{\sigma\sqrt{\theta}g-\sigma^2\frac{\theta}{2}} - Ke^{-r\theta}\big)\frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}dy \\
&= \int_{-\infty}^{d_2}\big(xe^{-\sigma\sqrt{\theta}g-\sigma^2\frac{\theta}{2}} - Ke^{-r\theta}\big)\frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}dy \\
&= \int_{-\infty}^{d_2}\big(xe^{-\sigma\sqrt{\theta}g-\sigma^2\frac{\theta}{2}} - \int_{-\infty}^{d_2}Ke^{-r\theta}\big)\frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}dy.
\end{aligned}
$$

Now, using the change of variable $z = y + \sigma\sqrt{\theta}$ we get

$$F(t,x) = xN(d_1) - Ke^{-r\theta}N(d_2),$$

where

$$N(d) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d} e^{-\frac{x^2}{2}}dx.$$

Similar calculations show that the price of the put option is

$$F(t,x) = Ke^{-r\theta}N(-d_2) - xN(-d_1).$$

CHAPTER 2

A PRODUCT SPACE

2.1 Introduction to the dual risk in unit-linked insurance products

In this chapter I will look at two different approaches in pricing guarantees in unit-linked insurance contracts. The unit-linked insurance contracts are very popular in many markets. The return obtained by the insured is linked to some financial index (or generally, the financial market). Some of these insurance contracts have also some kind of a death guarantee benefit.

We are therefore dealing with products that bear two different (independent) types of risk. First of all, we can look at the financial risk (related to the market). This risk was clearly stressed during the last few years, when the major stock market indices have dropped so much. On the other hand, the insurer deals with another type of risk, let's call it actuarial risk, related to the possibility of death for the insured (and hence the possibility of a claim). While the financial market model might be complete (any contingent claim is replicable by a trading strategy), the model that assumes both risks (financial and actuarial) is incomplete.

2.2 Product space

I will start by defining the two market models, the financial and the actuarial one and then I will take a look at the product market model.

2.2.1 Financial backgroud

The starting point for a mathematical model of the financial market was the paper by Bachelier [1], in 1900. He suggested that a possible approch in describing fluctuations in stock prices might be the Brownian motion. More than 60 years later, Samuelson [27], in 1965 proposed the idea that these fluctuations can better be described by a geometric Brownian motion, and this approach had the clear advantage that it didn't generate negative stock prices. This approach allowed Black and Scholes [4] (1973) and Merton [22] (1973) to determine the price of European options that doesn't allow arbitrage (no profits could arise from manipulations of stocks and options in any portfolio). The next important step is given by the work of Cox, Ross and Rubinstein [6] (1979) who investigated a simple discrete time model (binomial) in which the value of the stock between two trading times can only take two values. As limiting cases (by letting the length of time intervals between trading times tend to 0), they recovered (rediscovered) the option pricing formula by Black and Scholes.

2.2.2 Insurance - Actuarial background

The first known social welfare program with elements of life insurance appeared in the Roman Empire ("Collegia", AD 133). The first primitive mortality tables were published in 1662 by John Graunt and had only 7 age groups. The first complete mortality table is due to the astronomer Edmund Halley. The tables had been used for computations of premiums for life insurance contracts. De Moivre suggested methods for evaluations of life insurance products, combining interest and mortality under simple assumptions about mortality (which are used even today, like De Moivre Law). His assumption is basically the uniform death distribution between integral years and so this was an important step in describing the life insurance in a continuous time model (rather than discrete, up to that point). The modern utility theory, whose foundations were laid by Daniel Bernoulli, argues that risk should not be measured by expectations alone, because an important aspect is also the preference of the individual. For instance, it could be reasonable for a poorer individual to prefer an uncertain future wealth to another more unsure future wealth with a bigger expected value. This is very important in insurance in general, because it explains (together with the concept of pooling) why individuals prefer to buy insurance at a price which exceeds the expectations of future losses.

2.3 Combining the two markets

Let $(\Omega^f, \mathcal{F}^f, P^f)$ be a probability space. We equip this space with the filtration F^f of σ - algebras $\mathcal{F}_0^f \subset \mathcal{F}_1^f \subset \ldots \subset \mathcal{F}_T^f$ satisfying the usual conditions of right-continuity ($\mathcal{F}_t^f = \bigcap_{\epsilon>0} \mathcal{F}_{t+\epsilon}^f$) and completeness ($\mathcal{F}_0$ contains all P-negligible events in $\mathcal{F}$). We also take $\mathcal{F}_0^f = \{\emptyset, \Omega\}$ and $\mathcal{F}_T^f = \mathcal{F}^f$. Consider a d-dimensional process $X = (X^1, X^2, \ldots, X^d)$ which describes the evolution of the discounted prices of d tradable stocks. $X(\omega)$ is the path of X associated with $\omega \in \Omega^f$. A purely financial derivative is a random variable $H^f \in L^2(P^f, \mathcal{F}_T^f)$.

Let's consider now another filtered probability space $(\Omega^a, \mathcal{F}^a, F^a, P^a)$. The filtration is right-continuous but not necessarily complete. This space carries a pure insurance (actuarial) risk process which describes the development of insurance claims. An insurance risk process U is a random variable defined on $(\Omega^a, \mathcal{F}^a)$ and $U(\omega)$ is the path associated with $\omega \in \Omega^a$.

A pure insurance (actuarial) contract is a random variable $H^a \in L^2(P^a, \mathcal{F}_T^a)$.

We are next looking at the combined model $(\Omega, \mathcal{F}, F, P)$, which is defined as the product space of the two individual spaces, financial and actuarial. The construction of the combined model follows Moller [23]. We let $\Omega = \Omega^f \times \Omega^a$ and $P = P^f \otimes P^a$. Let's define the σ-algebra $\mathcal{N}$ generated by all subsets of null-sets from $\mathcal{F}^f \otimes \mathcal{F}^a$, that is:

$$\mathcal{N} = \sigma\{F \subseteq \Omega^f \times \Omega^a | \exists G \in \mathcal{F}^f \otimes \mathcal{F}^a : F \subseteq G, (P = P^f \otimes P^a)(G) = 0\}.$$

Next we define $\mathcal{F} = (\mathcal{F}^f \otimes \mathcal{F}^a) \vee \mathcal{N}$ and also the following filtrations on the product space (extensions of the original filtrations):

$$\mathcal{F}_t^1 = (\mathcal{F}_t^f \otimes \{\emptyset, \Omega^a\}) \vee \mathcal{N}, \qquad \mathcal{F}_t^2 = (\{\emptyset, \Omega^f\} \otimes \mathcal{F}_t^a) \vee \mathcal{N}.$$

Lemma 2.1 The filtrations defined above:

1. Satisfy the usual conditions;

2. They are independent;

3. The filtration $F = (\mathcal{F}_t)_{0\le t\le T}$ defined by $\mathcal{F}_t = \mathcal{F}_t^1 \vee \mathcal{F}_t^2$ satisfies the usual conditions. Moreover, $\mathcal{F}_t = (\mathcal{F}_t^f \otimes \mathcal{F}_t^a) \vee \mathcal{N}$.

Proof:

1. The completeness is trivial, as $\mathcal{N} \subseteq \mathcal{F}_t^1$ and $\mathcal{N} \subseteq \mathcal{F}_t^2$ $\forall t \in [0,T]$. Next we want to show that F^1 is right-continuous. We define, for $s \in [0,T]$,

$$\mathcal{D}_s = \{F_1 \times \Omega^a | F_1 \in \mathcal{F}_s^f\}.$$

By definition, $\sigma(\mathcal{D}_s) = \mathcal{F}_s^f \otimes \{\emptyset, \Omega^a\}$, and as $\mathcal{D}_s$ is also a σ-algebra, we get

$$\mathcal{D}_s = \mathcal{F}_s^f \otimes \{\emptyset, \Omega^a\}.$$

Hence

$$\bigcap_{\epsilon>0} \mathcal{F}_{t+\epsilon}^1 = \bigcap_{\epsilon>0} (\mathcal{D}_{t+\epsilon} \vee \mathcal{N}) = (\bigcap_{\epsilon>0} \mathcal{D}_{t+\epsilon}) \vee \mathcal{N} = \mathcal{D}_t \vee \mathcal{N},$$

where the last equality follows from the right-continuity of F^f. Similarly, one can prove the fact that F^2 is right-continuous (using the right-continuity of F^a.)

2. By definition, we need to show that $\forall\ F_1 \in \mathcal{F}_T^1$ and $\forall\ F_2 \in \mathcal{F}_T^2$, we have:

$$P(F_1 \cap F_2) = P(F_1) \cap P(F_2)$$

Let's consider at first $F_1 = F_1^f \times O_2$ and $F_2 = O_1 \times F_2^a$, where $F_1^f \in \mathcal{F}^f$, $F_2^a \in \mathcal{F}^a$ and $O_1 \in \{\emptyset, \Omega^f\}$, $O_2 \in \{\emptyset, \Omega^a\}$. Then,

$$P(F_1 \cap F_2) = P((F_1^f \times O_2) \cap (O_1 \times F_2^a)) = P((F_1^f \cap O_1) \times (F_2^a \cap O_2))$$

$$= P^f(F_1^f \cap O_1)P^a(F_2^a \cap O_2) = P^f(F_1^f)P^f(O_1)P^a(F_2^a)P^a(O_2)$$

$$= P(F_1)P(F_2)$$

Now, because the sets of this type generate the entire σ-algebras, this shows that they are independent.

3. A general result from probability theory shows that F satisfies the usual conditions and the equality is trivial.

2.4 Disjoint pricing techniques

Let's assume that the underlying assets (index) in the variable annuity contract follows the classical geometrical Brownian motion, described by the following differential equation under the physical measure P:

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

In the classical Black-Scholes model, it can be shown that there is an equivalent martingale measure (the risk-neutral probability measure Q) under which the price process follows the equation:

$$dS_t = rS_t dt + \sigma S_t d\tilde{W}_t,$$

where μ is the expected rate of return of the asset, σ is its standard deviation, r is the risk-free rate of interest (bank savings account rate) and W_t (and $\tilde{W}_t$ respectively) is a standard Brownian motion under P (and Q respectively). The difference between the two types of pricing is given by the expected rate of return of the asset under each probability measure (μ under the P-measure for the actuarial approach and r under the Q-measure for the financial approach). The expected loss at time t is in both cases:

$$V^P(t,T) = E_P[e^{-r(T-t)} max(K - S_T, 0)|\mathcal{F}_t],$$

and

$$V^Q(t,T) = E_Q[e^{-r(T-t)} max(K - S_T, 0)|\mathcal{F}_t].$$

The single premium at time 0 for each type of pricing is given by:

Premium (actuarial)=$\sum_{k=1}^{\omega-x} V^P(0,k)_k p_x q_{x+k}$, and

Premium (financial)=$\sum_{k=1}^{\omega-x} V^Q(0,k)_k p_x q_{x+k}$

The financial premium is a sum of Black-Scholes put prices. The only difference between the two formulas is that the risk-free rate in the financial price model is replaced by the expected return in the actuarial model. This leads to higher financial premium when $\mu > r$.

The financial pricing approach is meaningless in the absence of a hedging strategy. This might be seen as a disadvantage of the financial approach, which is yet counterbalanced by some clear advantages: the premium is independent of the expected rate of return of the underlying asset (while the actuarial premium could be affected by errors in its estimation) and the financial risk is eliminated by the hedging portfolio (strategy).

CHAPTER 3

RISK ANALYSIS

3.1 Derivatives in incomplete markets

There are several approaches for valuing and hedging derivatives in incomplete markets. This sections provides an overview of the most important techniques.

3.1.1 Super-replication

The idea of super-hedging (or super-replication) was first suggested by El Karoui and Quenez in 1995 [8]. In this case, there is no risk (for the hedger) associated with the derivative, as the super-hedging price is the smallest initial capital that allows the seller to construct a portfolio which dominates the payoff process of the derivative (option). El Karoui and Quenez showed that a super-hedging strategy exists provided that

$$\sup_{Q \in \mathcal{P}_e} E_Q(H) < \infty$$

where $\mathcal{P}_e$ is the set of all equivalent martingale measures and H is the claim. By defining the process

$$\bar{V}_t = ess.sup_{Q \in \mathcal{P}_e} E_Q[H|\mathcal{F}_t]$$

and deriving its decomposition of the form:

$$\bar{V}_t = \bar{V}_0 + \sum_{j=1}^{t} \bar{\varphi}_j \Delta S_j^\star - C_t$$

where C is increasing and $S_t^\star$ is the discounted price at time t, it can be shown that the initial capital that satisfies this condition is given by

$$V_0 = \sup_{Q \in \mathcal{P}_e} E_Q(H)$$

and V_0 is called the upper-hedging price of the claim H. Also, the super hedging strategy is determined by $\varphi_j = \bar{\varphi}_j$.

Moller [23] used this to compute the super-hedging price and strategy for unit-linked insurance products. Let us consider a living benefit contract, that pays $f(S_T)$ to survivors at time T from a group of l_x insured age x. If

$$P_t^f := E_Q\left[\frac{f(S_T)}{B_T}\middle|\mathcal{F}_t^f\right] = E_Q\left[\frac{f(S_T)}{B_T}\right] + \sum_{j=1}^{t} \alpha_j^f \Delta S_j^\star$$

is the no-arbitrage price of the purely financial contingent claim that pays $f(T)$ at time T, the cheapest self-financing super-hedging strategy is given by:

$$\vartheta_t = (l_x - N_{t-1})\alpha_t^f \tag{3.1}$$

and

$$\eta_t = l_x P_0^f + \sum_{j=1}^{t} \vartheta_j \Delta S_j^\star - \vartheta_t S_t^\star \tag{3.2}$$

and the price of the contract is $l_x \times P_0^f$. Hence, the super-hedging price is given by the number l_x of policies sold multiplied with the price of the financial contingent claim. Therefore, it assumes that no policy-holder dies until the expiration of the contract i.e. the survival probability to time T is 1.

3.1.2 Utility-based indifference pricing

The indifference premium is a price such that the optimal expected utility among all portfolios containing the prespecified number of options coincides with the optimal expected utility among all portfolios without options. In other words, the buyer (investor) is indifferent to including the option into the portfolio. This approach was first suggested by Hodges and Neuberger [16] and is now a standard concept to value European style derivatives in incomplete markets. Let us start by considering the so-called mean-variance utility function

$$u_\beta(Y) = E[Y] - a(Var[Y])^\beta,$$

where Y is the wealth at time T and u describes the insurance company's preferences (while a and β are constants). An insurance company with utility function u prefers the pair (P_β, H) (i.e. selling the contingent claim H for the premium P_β) to the pair $(\hat{P}_\beta, \hat{H})$ if

$$u_\beta(P_\beta - H) \geq u_\beta(\hat{P}_\beta - \hat{H}).$$

The indifference price $IP(H)$ for H is defined by

$$\underbrace{sup}_{\varphi:V_0(\varphi)=V_0} u_\beta(V_T(\varphi) + IP(H) - H) = \underbrace{sup}_{\bar{\varphi}:V_0(\bar{\varphi})=V_0} u_\beta(V_T(\bar{\varphi}))$$

where V_0 is the initial capital at time 0.

3.1.3 Quadratic approaches

These techniques can be divided into two groups: (local) risk-minimization approaches, proposed by Follmer and Sondermann [12] and mean-variance hedging approaches, proposed by Bauleau and Lamberton [2]. This approach has the big advantage that hedging strategies can be obtained quite explicitly.

3.1.4 Quantile hedging and shortfall risk minimization

In the quadratic approach, losses and gains are treated equally. This is not a desirable feature and the way out is given by *quantile hedging* [13] or *efficient hedging* [14]. The seller minimizes the expected shortfall risk subject to a given initial capital, i.e. the seller wants to minimize $E[l(H - V_T)^+]$ over all strategies φ, where $V_t = c + \int_0^t \varphi_u dS_u$ and l is the loss function ($l : \mathbf{R}_+ \longrightarrow \mathbf{R}_+$, increasing and convex with $l(0) = 0$). Although gains are not punished in this approach, they are not rewarded either.

3.2 Description of the GMDB problem

This chapter presents a methodology for pricing the guaranteed minimum death benefit of a variable annuity in a market model with jumps. Recent developments in the stock market make variable annuities very attractive products from the insured point of view, but less attractive for insurers. The insured still has the possibility of investment benefits, while avoiding the risk of a stock market collapse. The insurer wants to minimize its risk and yet sell a competitive product.

The financial market model consists of one riskless asset and one risky asset whose price jumps in proportions J at some random times τ which correspond to the jump times of a Poisson process. The model describes incomplete markets and there is no perfect hedging.

In the second part of the chapter, we describe a possible method of risk analysis for binomial tree models.

3.2.1 The model

In the Black-Scholes model, the share price is a continuous function of time. Some rare events (which are rather frequent lately), can accompany "jumps" in the share price. In this case the market model is incomplete, hence there is no perfect hedging of options.

We consider a market model with one riskless asset and one risky asset whose price jumps in proportions $J_1, J_2, \ldots, J_n, \ldots$ at some random times $\tau_1, \tau_2, \ldots, \tau_n, \ldots$ which correspond to the jump times of a Poisson process. Between the jumps the risky asset follows the Black-Scholes model.

The mathematical model consists of a probability space $(\Omega, \mathcal{F}, P)$, a Brownian motion

(W_t) and a Poisson process $(N_t)_{t\geq 0}$ with parameter λ. The jumps J_n are independent and identically distributed on $(-1,\infty)$ and $(F_t)_t$ is the filtration which incorporates all information available at time t. The price process (S_t) of the risky asset is described as follows:

On $[\tau_j, \tau_{j+1})$, $dS_t = S_t(\mu dt + \sigma dW_t)$ i.e. Black-Scholes model;

At time τ_j, the jump of (S_t) is given by $\Delta S_{\tau_j} = S_{\tau_j} - S_{\tau_j^-} = S_{\tau_{j-}} J_j$;

In other words, $S_{\tau_j} = S_{\tau_j^-}(1+J_j)$; As defined, (S_t) is a right-continuous process.

It is straightforward to see that we have the following formula for the price process:

$$S_t = S_0\left(\prod_{j=1}^{N_t}(1+J_j)\right)e^{(\mu-\frac{\sigma^2}{2})t+\sigma W_t} \tag{3.3}$$

A variable annuity is an investment wrapped with a life insurance contract. The convenient tax deferral characteristic of the variable annuities makes them a very interesting and popular investment and retirement instrument. The average age at which people buy their first variable annuity is 50. There are a few different types of GMDB options associated with variable annuities. The most popular are:

1. **Return of premium** - the death benefit is the larger of the account value on the date of death or the sum of premiums less partial withdrawals;

2. **Reset** - the death benefit is automatically reset to the current account value every x years;

3. **Roll-up** - the death benefit is the larger of the account value on the day of death or the accumulation of premiums less partial withdrawls accumulated at a specified interest rate (e.g. 1.5% in many 2003 contracts);

4. **Ratchet** (look back) - same as reset, except that the death benefit is not allowed

to decrease, except for withdrawals.

Let ω be the expiry date for a variable annuity with a return of premium GMDB option associated with (S_t). Let T be the random variable that models the future lifetime of the insured (buyer of the contract).Then the payoff of the product is:

$$P(T) = \begin{cases} H(T) & \text{if } T \leq \omega \\ S(\omega) & \text{if } T > \omega \end{cases}$$

where $H(t) = \max(S(0), S(t)) = S(t) + \max(S(0) - S(t), 0) = S(t) + (S(0) - S(t))_+$
Basically, the value of the guarantee at time 0 is given by the price of a put option with stochastic expiration date. It can be shown that in discrete settings and when the benefit is paid at the end of the year of death,

$$PV(GMDB) = \sum_{m=1}^{\omega-x} {}_{m-1|}q_x P(m, S_0) \tag{3.4}$$

where $P(m, S_0)$ is the price of the put option with expiry m and strike S_0, in the Black-Scholes model.

If the benefit is paid at the moment of death, then

$$PV(GMDB) = \int_0^\infty f_T(t) P(t, S_0) dt \tag{3.5}$$

where $f_T(t)$ is the pdf of the future lifetime random variable. Closed form expressions can be obtained for appropriate assumptions on T (constant force, UDD, Balducci etc).

Next we want to determine the price of the put option associated with GMDB in the market model described in the introduction, which minimizes the risk at maturity.

Suppose $E(J_1) < \infty$ and let $\widetilde{S}_t = e^{-rt}S_t$ for $s \leq t$. Then

$$E(\widetilde{S}_t|F_s) = \widetilde{S}_s E\Bigg(e^{(\mu - r - \frac{\sigma^2}{2})(t-s)+\sigma(W_t - W_s)} \prod_{j=N_s+1}^{N_t} \big((1+J_j)|F_s\big)\Bigg)$$

$$= \widetilde{S}_s E\Bigg(e^{(\mu - r - \frac{\sigma^2}{2})(t-s)+\sigma(W_t - W_s)} \prod_{j=1}^{N_t - N_s} (1+J_{N_s+j})\Bigg)$$

because $W_t - W_s$ and $N_t - N_s$ are independent of F_s.

Hence

$$E(\widetilde{S}_t|F_s) = \widetilde{S}_s e^{(\mu-r)(t-s)} E(\prod_{j=N_s+1}^{N_t} (1+J_j))$$

But

$$E(\prod_{j=N_s+1}^{N_t} (1+J_j)) = E(\prod_{j=1}^{N_t}(1+J_j)) - E(\prod_{j=1}^{N_t}(1+J_j))$$

and

$$E(\prod_{j=1}^{N_t}(1+J_j)) = \sum_{n=1}^{\infty} E(\prod_{j=1}^{n}(1+J_j))P(N_t = n)$$

$$= \sum_{n=1}^{\infty}(1+E(J_j))^n e^{-\lambda t}\frac{(\lambda t)^n}{n!} = \sum_{n=1}^{\infty} e^{-\lambda t}\frac{(\lambda t(1+E(J)))^n}{n!}$$

$$= e^{-\lambda t}e^{\lambda t(1+E(J))} = e^{\lambda t E(J)}$$

So

$$E(\widetilde{S}_t|F_s) = \widetilde{S}_s e^{(\mu-r)(t-s)} e^{\lambda(t-s)E(J)}$$

Hence $(\widetilde{S}_t)$ is a martingale iff $\mu = r - \lambda E(J)$. In our case, we want to price a put option with strike S_0 and expiry T.

Let $f(x) = (S_0 - x)_x$. The price of the put option which minimizes the risk at time t is given by:

$$E(e^{-r(T-t)}f(S_t)|F_t) = E\Bigg(e^{-r(T-t)} f\Bigg(S_t e^{(\mu - r - \frac{\sigma^2}{2})(t-s)+\sigma(W_t - W_s)} \prod_{j=N_t+1}^{N_T} (1+J_j)\Bigg)\Bigg|F_t\Bigg)$$

$$= E\Bigg(e^{-r(T-t)} f\Bigg(S_t e^{(\mu - r - \frac{\sigma^2}{2})(t-s)+\sigma(W_t - W_s)} \prod_{j=1}^{N_{T-t}} (1+J_j)\Bigg)\Bigg)$$

$$= E\Bigg(P\Bigg(t, S_t e^{-\lambda(T-t)E(J)} \prod_{j=1}^{N_{T-t}} (1+J_j)\Bigg)\Bigg)$$

where $P(t,x)$ is the function that gives the price of the option for the Black-Scholes model. As N_{T-t} is Poisson with parameter $\lambda(T-t)$,

$$E(e^{-r(T-t)} f(S_t)|F_t) = \sum_{n=0}^{\infty} E\Bigg(P\Bigg(t, S_t e^{-\lambda(T-t)E(J)} \prod_{j=1}^{n}(1+J_j)\Bigg)\Bigg) \frac{e^{-\lambda(T-t)}\lambda^n (T-t)^n}{n!}$$

Let us now assume that J takes values in $\{u, d\}$ and $P(J = u) = p$, $P(J = d) = 1-p$. We will use the following:

Lemma 1: Let N be Poisson with parameter λ.

Let $S = \sum_{n=1}^{N} V_n$ with $P(V_n = u) = p$, and $P(V_n = d) = 1-p$. Then law(S) =law($uN_1 + dN_2$), where N_1 is Poisson λp and N_2 is Poisson ($\lambda(1-p)$).

Proof: One method would be to show that the two random variables have the same moment generating function.

Another method would be to re-write $S = \sum_{n=1}^{N}(u + (d-u)I_n)$, where $I_n = 0$ with probability p and $I_n = 1$ with probability $1-p$. So,

$$S = uN + (d-u)\sum_{n=1}^{N} I_n = uN_1 + dN_2,$$

because $\sum_{n=1}^{N} I_n$ is Poisson ($\lambda(1-p)$). This completes the proof of the lemma.

Now,

$$\prod_{j=1}^{N_{T-t}} (1+J_j) = \prod_{j=1}^{N_{T-t}} e^{ln(1+J_j)} = e^{\sum_{j=1}^{N_{T-t}} ln(1+J_j)},$$

and using the lemma we have $\sum_{j=1}^{N_{T-t}} ln(1+J_j)$ has the same law as

$ln(1+u)N_1 + ln(1+d)N_2$ where N_1 and N_2 are iid with parameters λp and $\lambda(1-p)$

respectively.

So, the price of the option at time t is given by:

$$\underbrace{\sum_{n_1,n_2}\sum_{k_1,k_2}}_{\alpha k_1+\beta k_2=\alpha n_1+\beta n_2} P\left(t, S_0 e^{-\lambda(T-t)[pu+(1-p)d]} e^{ln(1+u)n_1+ln(1+d)n_2}\right) e^{-\lambda} \frac{\lambda^{k_1+k_2} p^{k_1}(1-p)^{k_2}}{(k_1)!(k_2)!} \tag{3.6}$$

where $\alpha = ln(1+u)$ and $\beta = ln(1+d)$.

Replacing now the price of the put option in formula (3.2) we get the price for GMDB paid at the end of the year of death or in formula (3.3) we get the price of the GMDB for continuous time model, with benefit paid at the moment of death.

Most of the time, α and β are linearly independent over $\mathbf{Z}$, so in this case the decomposition $\alpha n_1 + \beta n_2$ is unique, and the price of the option is given by:

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} P\left(t, S_0 e^{-\lambda(T-t)[pu+(1-p)d]} e^{ln(1+u)n_1+ln(1+d)n_2}\right) e^{-\lambda} \frac{\lambda^{n_1+n_2} p^{n_1}(1-p)^{n_2}}{(n_1)!(n_2)!} \tag{3.7}$$

3.2.2 Other market models with jumps

The problem of the price jumps can be analyzed in other models too. Another model could be described as follows: only one jump whose time occurance is uniformly distributed on the contract length. Let ω be the expiration date of the contract and T_j the random variable modeling the time of occurance of the jump. Let also T_d be the random variable that models the lifetime of the insurer. Let's assume for simplicity that T_d is exponential, i.e. $f_{T_d} = \lambda e^{-\lambda t}$.

The probability that the jump occurs before the death is

$$P(T_j < T_d) = \int_0^{\omega}\int_{t_j}^{\infty} \frac{1}{\omega}\lambda e^{-\lambda t_d} dt_d dt_j = \int_0^{\omega} \frac{1}{\omega} e^{-\lambda t_j} dt_j =$$

$$= \frac{1}{\omega} \frac{e^{-\lambda t_j}}{-\lambda} \Big|_0^{\omega} = \frac{1 - e^{-\lambda \omega}}{\lambda \omega}$$

Let τ be the random time of the jump. Then,

$$\begin{cases} S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t} & \text{for } t < \tau \text{ and} \\ S_t = S_0 (1 + J) e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t} & \text{for } t \geq \tau \end{cases}$$

As in the first model, the discounted price process is a martingale for specific jump processes and the GMDB price can be found similarly.

3.2.3 Risk analysis

We focus our attention now on a binomial tree model, and for simplicity we will assume that the price process (S_t) of a risky asset follows a simple random walk, going up one unit with probability 1/2 and down one unit with probability 1/2. For simplicity we will assume that $S_0 = 0$, using a translation of the random variable that models the stock price. Let $\tau_N = inf\{k \geq 0 : |S_k| = N\}$ be the first time the random walk is at distance N from the origin. If we think about the stock price, τ_N is the random time when S_n goes up or down N units, for the first time. Hence, τ_N can be interpreted as a measure of risk.

First, it is quite easy to show that the distribution of τ_N has an exponential tail and hence has moments of all orders. Let $T_N = inf\{k \geq 0 : S_k = N\}$. If $\omega \in \{\tau_N = n\}$, then $\omega \in \{T_N = n\} \cup \{T_{-N} = n\}$.

So $P(\tau_N = n) \leq P(T_N = n) + P(T_{-N} = n) \underbrace{=}_{symmetry} 2P(T_N = n)$.

So, $P(\tau_N = n) \leq 2P(T_N = n)$. Next we want to find $P(T_N = n)$. To get a path that gets to N for the first time at $t = n$, we need the path to be at $N - 1$ at time $t = n - 1$ (see Figure 3.1). Hence we need to count all possible paths that are at $N - 1$ at time $n - 1$ and which never rise above $N - 1$ before time $n - 1$.

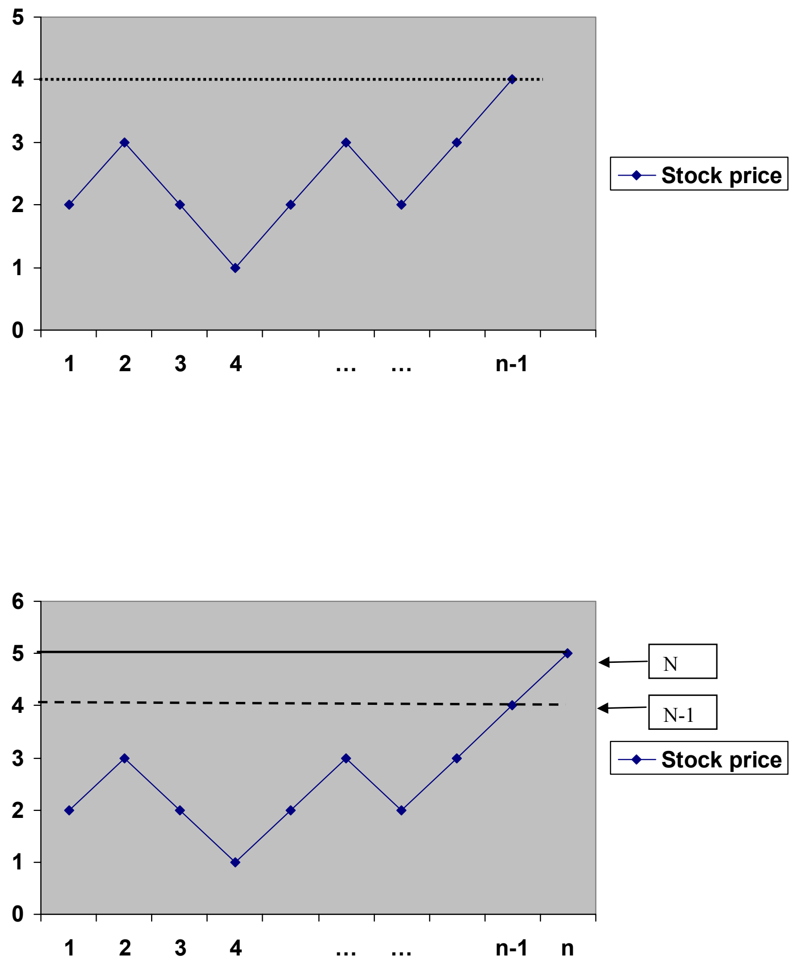

Figure 3.1: Random walk for stock price

Generally, for $r \geq k$ and using the reflection principle, the probability of a path from $(0,0)$ to (n,k) with maximum $\geq r$ equals the probability of a path from $(0,0)$ to $(n, 2r-k)$. Let us denote this probability by $P_{n,2r-k}$. We have

$$P(\text{path with max} \leq r) = P(\text{path with max } < r) + P(\text{path with max } = r)$$

So,

$$P(\text{path with max } = r) = P(\text{path with max} \leq r) - P(\text{path with max} \leq r-1)$$

$$= 1 - P(\text{path with max} \geq r+1) - 1 + P(\text{path with max} \geq r)$$

$$= P_{n,2r-k} - P_{n,2r+2-k}$$

So the probability of a path from $(0,0)$ to (n,k) with maximum r equals $= P_{n,2r-k} - P_{n,2r+2-k}$ and hence the probability of a path from $(0,0)$ to $(n-1, N-1)$ with maximum $N-1$ is $= P_{n-1,N-1} - P_{n-1,N+1}$. Therefore

$$P\big(T_N = n\big) = \frac{1}{2}\big(P_{n-1,N-1} - P_{n-1,N+1}\big).$$

But clearly,

$$P_{n,k} = P(S_n = k) = C^n_{\frac{n+k}{2}} 2^{-n}$$

as you need $\frac{n+k}{2}$ steps up $(+1)$ and $\frac{n-k}{2}$ down (-1), each with probability $\frac{1}{2}$. Then we have

$$P_{n-1,N-1} - P_{n-1,N+1} = C^{n-1}_{\frac{n+N}{2}-1} 2^{-n+1} - C^{n-1}_{\frac{n+N}{2}} 2^{-n+1}.$$

Let $k = \frac{n+N}{2}$. We get

$$P_{n-1,N-1} - P_{n-1,N+1} = C^{n-1}_{k-1} 2^{-n+1} - C^{n-1}_{k} 2^{-n+1}.$$

$$= \frac{(n-1)!}{(n-k)!(k-1)!} 2^{-n+1} - \frac{(n-1)!}{(n-k-1)!k!} 2^{-n+1}$$

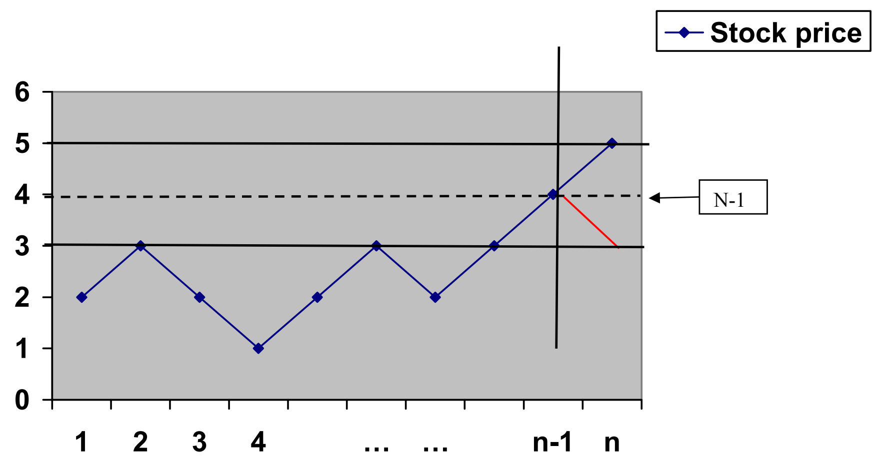

Figure 3.2: Reflection principle for random walks

$$= 2^{-n+1}\left[\frac{n!k}{(n-k)!k!n} - \frac{n!(n-k)}{(n-k)!k!n}\right]$$

$$= \frac{2^{-n+1}}{n}\frac{n!}{(n-k)!k!}(2k-n) = \frac{N}{n}C^{n}_{\frac{n+N}{2}}2^{-n+1},$$

as $N = 2k - n$. So

$$P(T_N = n) = \frac{N}{n}C^{n}_{\frac{n+N}{2}}2^{-n}.$$

Recall Stirling's formula

$$n! \sim \sqrt{2\pi n}n^n e^{-n}, \text{ as } n \to \infty$$

So,

$$P\big(T_N = n\big) = \frac{N}{n}\frac{n!}{\big(\frac{n+N}{2}\big)!\big(\frac{n-N}{2}\big)!}2^{-n}$$

$$\sim \frac{N}{n}\frac{n^n e^{-n}\sqrt{2\pi n}}{\big(\frac{n+N}{2}\big)^{\frac{n+N}{2}}\big(\frac{n-N}{2}\big)^{\frac{n-N}{2}}e^{-\frac{n+N+n-N}{2}}\sqrt{2\pi\frac{n+N}{2}}\sqrt{2\pi\frac{n-N}{2}}}2^{-n}$$

$$\sim \frac{N}{n}\frac{1}{\big(\frac{n+N}{2}\big)^{\frac{n+N}{2}}\big(\frac{n-N}{2}\big)^{\frac{n-N}{2}}}\frac{1}{\sqrt{2\pi n}}\frac{1}{\sqrt{\frac{n+N}{2n}\frac{n-N}{2n}}}2^{-n}$$

$$\sim \frac{N}{n}\big(1+\frac{N}{n}\big)^{-\frac{n+N}{2}}\big(1-\frac{N}{n}\big)^{-\frac{n-N}{2}}2^n 2^{-n}\frac{1}{\sqrt{2\pi n}}2\big(1+\frac{N}{n}\big)^{-\frac{1}{2}}\big(1-\frac{N}{n}\big)^{-\frac{1}{2}}$$

$$\sim \sqrt{\frac{2}{\pi n}}\frac{N}{n}\big(1-\frac{N^2}{n^2}\big)^{-\frac{n}{2}}\big(1+\frac{N}{n}\big)^{-\frac{N}{2}}\big(1-\frac{N}{n}\big)^{\frac{N}{2}}\big(1-\frac{N^2}{n^2}\big)^{-\frac{1}{2}}.$$

Let us now consider the following setting:

$$\frac{N}{\sqrt{n}} \to x \text{ and } \frac{N}{n} \to 0.$$

Then we have

$$\big(1-\frac{N^2}{n^2}\big)^{-\frac{n}{2}} = \left[\big(1-\frac{x^2}{n}\big)^{-n}\right]^{\frac{1}{2}} \sim e^{\frac{x^2}{2}}$$

$$\big(1+\frac{N}{n}\big)^{-\frac{N}{2}} = \left[\big(1+\frac{x}{\sqrt{n}}\big)^{\sqrt{n}}\right]^{-\frac{x}{2}} \sim e^{-\frac{x^2}{2}}$$

$$\big(1-\frac{N}{n}\big)^{\frac{N}{2}} = \left[\big(1-\frac{x}{\sqrt{n}}\big)^{-\sqrt{n}}\right]^{-\frac{x}{2}} \sim e^{-\frac{x^2}{2}}$$

So,

$$P(T_N = n) = \sqrt{\frac{2}{\pi}} \frac{N}{\sqrt{n^3}} e^{-\frac{N^2}{2n}}$$

Next we want to look directly at $P(\tau_N = n)$. Let $k \in N$ be fixed, and $l \in N$ such that $-k \le -l \le l \le k$.

Let y_l^n be the probability of a path from $(0,0)$ to $(n, \pm l)$ without passing through $\pm k$.

Note: (a, b) means getting to the value b at time a.

We have the following recurrence relations:

$$\begin{cases} y_0^n = \frac{1}{2} y_1^{n-1} \\ y_1^n = y_0^{n-1} + \frac{1}{2} y_2^{n-1} \\ \dots \\ y_l^n = \frac{1}{2} y_{l-1}^{n-1} + \frac{1}{2} y_{l+1}^{n-1}, \quad \text{for } 2 \le l \le k-2 \\ \dots \\ y_{k-1}^n = \frac{1}{2} y_{k-2}^{n-1} \\ y_k^n = \frac{1}{2} y_{k-1}^{n-1} \end{cases}$$

We want to find $p_n := y_k^n = \frac{1}{2} y_{k-1}^{n-1}$. Then we will take k=N and get the distribution of τ_N.

Let $\begin{pmatrix} y_0^n \\ y_1^n \\ . \\ . \\ . \\ y_{k-1}^n \end{pmatrix}$ be a column vector.

The recurrence relations can be written as:

$$y^n = A y^{n-1}$$

where A is a (k)x(k) matrix:

$$A = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \frac{1}{2} & 0 & \dots & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & \frac{1}{2} & 0 \end{pmatrix}$$

Let $P_A(t) = det(tI - A)$. Let $B_k := tI - A$ and let $P_k := det(B_k)$. Then, using the last row of matrix B_k,

$$detB_k = det \begin{pmatrix} t & -\frac{1}{2} & 0 & \dots & 0 \\ -1 & t & -\frac{1}{2} & \dots & 0 \\ 0 & -\frac{1}{2} & t & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & -\frac{1}{2} & t \end{pmatrix} = -(-\frac{1}{2})det(C) + tdet(B_{k-1})$$

Then again, using the last row of matrix C,

$$det(C) = -(-\frac{1}{2})det(D) + (-\frac{1}{2})det(B_{k-2}) = \frac{1}{2}det(D) - \frac{1}{2}det(P_{k-2}).$$

But D has the last column 0, so $det(D) = 0$. Hence $det(C) = -\frac{1}{2}P_{k-2}$ and so

$$P_k = -\frac{1}{4}P_{k-2} + tP_{k-1} \tag{3.8}$$

The recurrence relation $P_k - tP_{k-1} + \frac{1}{4}P_{k-2} = 0$ has characteristic polynomial $x^2 - tx + \frac{1}{4}$. The roots for this polynomial are $x_{1,2} = \frac{t \pm \sqrt{t^2-1}}{2}$ and so,

$$P_k = \alpha_1 x_1^k + \alpha_2 x_2^k.$$

But $P_1 = t$, because $A = 0$ when $k = 1$ and $P_2 = t^2 - \frac{1}{2}$.

So we can identify $\alpha_1 = \alpha_2 = 1$. Hence:

$$P_k = x_1^k + x_2^k = \frac{1}{2^k}\Big((t + \sqrt{t^2-1})^k + (t - \sqrt{t^2-1})^k\Big).$$

Let $P(t) = (t + \sqrt{t^2-1})^k + (t - \sqrt{t^2-1})^k = 2^k P_k = 2^k P_A(t)$.

Then $P(A) = 2^k P_A(A) = 0$, by Cayley's theorem. If $P(t) = a_0 t^k + \cdots + a_k$, then

$$a_0 A^k + \cdots + a_k I = 0. \tag{3.9}$$

Next, let's multiply (5) to the right by y^{n-k}, which is a column vector, for $n \geq k$. We get

$$a_0y^n + a_1y^{n-1} + \cdots + a_ky^{n-k} = 0$$

In particular, if we read only the last line we get:

$$a_0y^n_{k-1} + a_1y^{n-1}_{k-1} + \cdots + a_ky^{n-k}_{k-1} = 0,\ \forall\ n \geq k.$$

But $p_n = \frac{1}{2}y^{n-1}_{k-1}$, so we get the recurrence:

$$a_0p_n + a_1p_{n-1} + \cdots + a_kp_{n-k} = 0,\ \text{for } n \geq k+1$$

Let now $Q(t) = a_0 + \cdots + a_kt^k$. Consider also the power series $S(t) = p_0 + p_1t + p_2t^2 + \ldots$

Let $Q(t)S(t) = c_0 + c_1t + c_2t^2 + \ldots$

For $n \leq k$, $c_n = a_0p_n + a_1p_{n-1} + \cdots + a_np_0$.

For $n \geq k+1$, $c_n = a_0p_n + a_1p_{n-1} + \cdots + a_kp_{n-k} = 0$.

But as $p_0 = p_1 = \cdots = p_{k-1} = 0$, we get that $c_0 = c_1 = \cdots = c_{k-1} = 0$ and $c_k = a_0p_k$.

Hence $Q(t)S(t) = c_kt^k = a_0p_kt^k$ and so

$$S(t) = \frac{a_0p_kt^k}{Q(t)} \tag{3.10}$$

We have:

$Q(t) = a_0 + a_1t + \cdots + a_kt^k$, and

$P(t) = a_0t^k + \cdots + a_k$.

These two polynomials are reciprocal and

$$Q(t) = t^kP(\frac{1}{t}) = t^k\left(\left(\frac{1}{t} + \sqrt{\frac{1}{t^2} - 1}\ \right)^k + \left(\frac{1}{t} - \sqrt{\frac{1}{t^2} - 1}\ \right)^k\right)$$

$$(1 + \sqrt{1-t^2})^k + (1 - \sqrt{1-t^2})^k.$$

In particular, $a_0 = Q(0) = 2^k$. Also, $p_k = \frac{1}{2^{k-1}}$ (one gets to $\pm k$ after k steps iff there are $k\ +1's$ or $k\ -1's$, and any of these two events happen with probability $\frac{1}{2^k}$, hence

$p_k = 2\frac{1}{2^k}$).

We then get $a_0 p_k = 2$, so

$$S(t) = \frac{a_0 p_k t^k}{Q(t)} = \frac{2t^k}{(1+\sqrt{1-t^2})^k + (1-\sqrt{1-t^2})^k}.$$

As a conclusion, $P(\tau_N = n)$ is the coefficient of t^n in the Taylor series of

$$S(t) = \frac{2t^N}{(1+\sqrt{1-t^2})^N + (1-\sqrt{1-t^2})^N} \tag{3.11}$$

3.3 Game options in incomplete markets

The game options are contracts which enable both the buyer and seller to stop them at any time up to maturity, when the contract is terminated anyway. An example is the Israeli call option, which is an American style call option with strike price K where the seller can also terminate the contract, but at the expense of a penalty $\delta_{t_i} \geq 0$.

To define the game contingent claim precisely, let $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\in[0,T]})$ be a filtered probability space satisfying the usual conditions of right-continuity and completeness, and let $(U_{t_i})_{i=0,\ldots,k}$, $(L_{t_i})_{i=0,\ldots,k}$, $(M_{t_i})_{i=0,\ldots,k}$ be sequences of real-valued random variables adapted to $(\mathcal{F}_{t_i})_{i=0,\ldots,k}$ with $L_{t_i} \leq M_{t_i} \leq U_{t_i}$ for $i = 0, \ldots, k-1$ and $L_{t_k} = M_{t_k} = U_{t_i}$. If A terminates the contract at time t_i before B exercises then A should pay B the amount U_{t_i}. Similarly, if B terminates the contract at time t_i before A exercises then B should pay A only the amount L_{t_i}. Finally, if A terminates and B exercises at the same time, then A pays B the amount M_{t_i}.

Let $\mathcal{S}_i$, $i = o, \ldots, k$ be the set of stopping times with values in $t_i, \ldots, t_k$.

For instance, if A terminates the contract at the random time $\sigma \in \mathcal{S}_0$ and B exercises at the random time $\tau \in \mathcal{S}_0$, then A will pay B at the random time $\sigma \wedge \tau$ the amount

$$R(\sigma, \tau) = U_\sigma I(\sigma < \tau) + L_\tau I(\tau < \sigma) + M_\tau I(\tau = \sigma).$$

Example: In the case of the Israeli call option, $L_{t_i} = (S^1_{t_i} - K)^+$, $U_{t_i} = (S^1_{t_i} - K)^+ + \delta_{t_i}$ and $M_{t_i} = (S^1_{t_i} - K)^+ + (\delta_{t_i})/2$. The game version of an American option is cheaper, because it is a safer investment for the company that sells it. In the case of a complete market model, the seller A wants to minimize $E_Q(R(\sigma, \tau))$ and the buyer B wants to maximize the same quantity, where Q is the unique equivalent martingale measure. This is equivalent to a zero-sum Dynkin stopping game, which has a unique value, which is also the unique no-arbitrage price of the game option [20]. In incomplete markets, this approach fails because there is more than one equivalent martingale measure. A possible approach was suggested by Christoph Kuhn [9], in his Ph. D. thesis and is based on utility maximization.

We consider $u_1, u_2 : \mathbf{R} \longrightarrow \mathbf{R}$ two nondecreasing and concave functions that correspond to the utility functions of the seller respectively the buyer of the option. If we use the game theory language, each player chooses a stopping time $\sigma \in \mathcal{S}_0$ (respectively $\tau \in \mathcal{S}_0$ and a trading strategy ϑ. The seller wants to maximize

$$E_P\big(u_1\big(C_1 - R(\sigma,\tau) + \int_0^T \vartheta dS_t\big)\big),$$

while the buyer wants to maximize

$$E_P\big(u_2\big(C_2 + R(\sigma,\tau) + \int_0^T \vartheta dS_t\big)\big),$$

where the random variable $C_i \in \mathcal{F}_T$ is the exogenous endowment of the i-th player.

Definition: We say that a pair $(\sigma^*,\tau^*) \in \mathcal{S}_0 \times \mathcal{S}_0$ is a Nash (or non-cooperative) equilibrium point, if for all $(\sigma,\tau) \in \mathcal{S}_0 \times \mathcal{S}_0$,

$$\sup_\vartheta E_P\big(u_1\big(C_1 - R(\sigma^*,\tau^*) + \int_0^T \vartheta dS_t\big)\big) \geq \sup_\vartheta E_P\big(u_1\big(C_1 - R(\sigma,\tau^*) + \int_0^T \vartheta dS_t\big)\big),$$

and

$$\sup_\vartheta E_P\big(u_2\big(C_2 + R(\sigma^*,\tau^*) + \int_0^T \vartheta dS_t\big)\big) \geq \sup_\vartheta E_P\big(u_2\big(C_2 - R(\sigma^*,\tau) + \int_0^T \vartheta dS_t\big)\big).$$

In the case of exponential utility ($u(x) = 1 - e^{-\alpha_1 x}$) Nash equilibrium can be constructed for various trading strategies. But there are also cases (e.g. logarithmic utility function) when no Nash equilibrium exists.

Game options might be interesting in an insurance environment too. If the seller of a variable annuity considers that the product becomes of high risk (for instance the death benefit is much bigger than the account value), then the insurance company can terminate the contract and be better off with the penalty than with the risk of a huge claim.

3.4 Hedging insurance claims in incomplete markets

We will use the mean-variance hedging approaches proposed by Follmer and Sondermann [12] and the work with equity-linked insurance contracts by Moller [23] . Consider a financial market with 2 traded assets: a stock with stochastic price process S and a bond with deterministic price process B. The financial market model is given by $(\Omega^f, \mathcal{F}^f, P^f)$ and the price processes follow:

$$dS_t = \alpha(t, S_t)S_t dt + \sigma(t, S_t)S_t dW_t$$

$$dB_t = r(t, S_t)B_t dt,$$

where $(W_t)_{0\leq t\leq T}$ is a standard Brownian motion on $[0, T]$. The probability space is equipped with a filtration F^f satisfying the usual conditions, defined by $\mathcal{F}_t^f = \sigma\{(S_u, B_u), u \leq t\} = \sigma\{S_u, u \leq t\}$. We can also define in this setting the market price of risk associated with S, $\nu_t = \frac{\alpha_t - r_t}{\sigma_t}$. In the Black-Scholes setting, the price processes are given by

$$S_t = S_0 exp((\alpha - \frac{1}{2}\sigma^2)t + \sigma W_t)$$

$$B_t = exp(rt)$$

We say that two probability measures P and $P^\star$ are equivalent iff they have the same null-sets. By definition, the following probability measure $P^\star$ defined by

$$\frac{dP^\star}{dP} = exp\Big(- \int_0^T \frac{\alpha_u - r_u}{\sigma_u} dW_u - \frac{1}{2}\int_0^T (\frac{\alpha_u - r_u}{\sigma_u})^2 dW_u \Big) \equiv U_T$$

is equivalent with P and $S^\star := S_t B_t^{-1}$ is a $P^\star$-martingale.

A trading strategy (or dynamic portfolio) is a 2-dimensional process $\varphi_t = (\vartheta_t, \eta_t)$ satisfying certain integrability conditions (indicated later) and where ϑ is predictable

(ϑ_t is $\mathcal{F}^f_{t-1}$-measurable, and η is adapted to F^f. The pair $\varphi_t = (\vartheta_t, \eta_t)$ is the portfolio held at time t (ϑ_t is the number of shares of the stock held at time t and η_t is the discounted amount invested in the savings account).

Thus, the value process is given by $\widehat{V}^{\varphi}_t = \vartheta_t S_t + \eta_t B_t$.

The trading strategy is self-financing if $\widehat{V}^{\varphi}_t = \widehat{V}^{\varphi}_0 + \int_0^t \vartheta_u dS_u + \int_0^t \eta_u dB_u, \; \forall \; 0 \leq t \leq T$.

A contingent claim with maturity T is a random variable X that is $\mathcal{F}^f_T$-measurable and $P^{\star}$-square integrable. The contingent claim is just a simple claim when $X = g(S_t)$, where $g : \mathbf{R}_+ \longrightarrow \mathbf{R}$. The contingent claim is attainable if $\exists \varphi \; s.t. \; \widehat{V}^{\varphi}_t = X \;\; P - a.s.$ If any contingent claim is attainable, the market is called *complete*.

The financial market is now complemented with an insurance portfolio. The assumption in the insurance market model are that the lifetimes of the individuals are independent and identically distributed. We will denote by l_x the number of persons of age x in the group. The probability space $(\Omega^a, \mathcal{F}^a, P^a)$ describes the insurance model. The remaining lifetimes are modelled by the random variables $T_1, T_2, \ldots, T_{l_x}$, which are iid and non-negative. The hazard function is μ_{x+t} and the survival function is given by ${}_tp_x := P^a(T_i > t) = exp\big(- \int_0^t \mu_{x+\tau} d\tau \big)$. Next we define a uni-variate process $N = (N_t)_{0 \leq t \leq T}$ describing the number of deaths in the group, by:

$$N_t = \sum_{i=1}^{l_x} I(T_i \leq t).$$

This process is a cadlag. We can now equip the probability space with a filtration F^a, by $\mathcal{F}^a_t = \sigma\{N_u, u \leq t\}$ The stochastic intensity of the counting process N can be describes as follows:

$$E[dN_t | \mathcal{F}^a_t] = (l_x - N_{t^-}) \mu_{x+t} dt \equiv \lambda_t dt$$

3.5 The combined model in the GMDB case

We define the product space (financial × actuarial) as we did in the general case in chapter 2. The filtration in the combined model is given by $F = (\mathcal{F}_t)_{0\leq t\leq T}$, where $\mathcal{F}_t = \mathcal{F}_t^f \vee \mathcal{F}_t^a$. Let's say the contract between insured and insurer has the individual liability $g_t = g(t, S_t)$ at time t. Overall, (for the entire portfolio), the insurer's liability is:

$$\begin{aligned} H_T &= B_T^{-1} \sum_{i=1}^{l_x} g(T_i, S_{T_i}) B_{T_i}^{-1} B_T I(T_i \leq T) \\ &= \sum_{i=1}^{l_x} \int_0^T g(u, S_u) B_u^{-1} dI(T_i \leq u) \end{aligned}$$

which can be written with respect to the counting process N as follows:

$$H_T = \int_0^T g(u, S_u) B_u^{-1} dN_u.$$

The equivalent martingale measure is not unique anymore, but we'll only use $P^\star$ defined above, which is known as the *minimal martingale measure*, cf. Schweizer [28] (1991).

We introduce the deflated value process V^φ by

$$V_t^\varphi = \widehat{V}_t^\varphi B_t^{-1} = \vartheta_t S_t^\star + \eta_t$$

and the space $\mathcal{L}^2(P_S^\star)$ of F-predictable processes ϑ satisfying

$$E^\star\left[\int_0^T (\vartheta)^2 d\langle S^\star\rangle_u\right] < \infty.$$

I this setting, an F-trading strategy is a process $\varphi_t = (\vartheta_t, \eta_t)$ with $\vartheta \in \mathcal{L}^2(P_S^\star)$ and η F-adapted with V^φ cadlag and $V_t^\varphi \in \mathcal{L}^2(P_S^\star)$ $\forall t$.

Definition 3.5.1 (Schweizer [29], 1994). The *cost process* associated with the strategy

φ is defined by

$$C_t^\varphi = V_t^\varphi - \int_0^t \vartheta_u dS_u^\star,$$

and the *risk process* associated with the strategy φ is defined by

$$R_t^\varphi = E^\star[(C_T^\varphi - C_t^\varphi)^2|\mathcal{F}_t].$$

A few comments about these two processes are very interesting. First of all, the initial cost of the portfolio is $C_0^\varphi = V_0^\varphi$ and it is tipically greater than zero, except for cases when we start the portfolio with some short sales. C_t^φ, the total cost incured in $[0,t]$ can be seen as an initial cost and the cost during $(0,t]$.

A strategy is called *mean-self-financing* if the cost process C^φ is a $(F, P^\star)$-martingale. In particular, the strategy $\varphi = (\vartheta, \eta)$ is self-financing if and only if

$$V_t^\varphi = V_0^\varphi + \int_0^t \vartheta_u dS_u^\star,$$

or, in other words, if and only if the only cost associated with φ is the initial cost $(C_t^\varphi = C_0^\varphi = V_0^\varphi,\ P^\star - a.s.)$

We have seen that the combined model is not complete, and hence there are contingent claims which cannot be replicated by self-financing trading strategies. We will consider the next best thing, i.e. we are looking for strategies that are able to generate the contingent claim at time T, but only at some cost defined by C_T^φ. Let H be a $\mathcal{F}_T$-measurable random variable (the contingent claim) and we are looking for a strategy φ with $V_T^\varphi = H\ a.s.$ and cost process C^φ. Note that the cost is not known at time 0 (unless the strategy is self-financing).

The mean squared error is defined as the value of the risk process at time 0. Hence, as $\mathcal{F}_0$ is trivial

$$R_0^\varphi = E^\star\big[(C_T^\varphi - C_0^\varphi)^2|\mathcal{F}_0\big] = E^\star\big[(C_T^\varphi - C_0^\varphi)^2\big] = E^\star\left[\left(H - \int_0^T \vartheta_u dS_u^\star - C_0^\varphi\right)^2\right]$$

Thus, R_0^φ is minimized for $C_0^\varphi = E^\star[H]$. But $E^\star[H] = E^\star[C_T^\varphi]$ and so the trading strategy should be chosen such that ϑ minimizes the variances $E^\star\big[(C_T^\varphi - E^\star[C_T^\varphi])^2\big]$. The strategy will not be unique (there is an entire class of strategies minimizing the mean squared error).

The construction of the strategies follows Follmer and Sondermann [12]. First we define the *intrinsic process* $V^\star$ by $V_t^\star = E^\star[H|\mathcal{F}_t]$. Next, using the Galtchouk-Kunita-Watanabe decomposition (see Appendix), we can decompose $V_t^\star$ uniquely as follows:

$$V_t^\star = E^\star[H] + \int_0^t \vartheta_u^H dS_u^\star + L_t^H,$$

where L^H is a zero-mean $(F, P^\star)$-martingale, L^H and $S^\star$ are orthogonal and ϑ^H is a predictable process in $\mathcal{L}^2(P_S^\star)$. Follmer and Sondermann [12] proved that

Theorem 3.5.1.: An admissible strategy $\varphi_t = (\vartheta_t, \eta_t)$ has minimal variance

$$E^\star\big[(C_T^\varphi - E^\star[C_T^\varphi])^2\big] = E^\star[(L_T^H)^2]$$

if and only if $\vartheta = \vartheta^H$.

The number of bonds held at time 0 is given by $\eta_0 = E^\star[H] - \vartheta_0 S_0^\star$. Follmer and Sondermann [12] have refined the process and have found an admissible strategy minimizing the risk process R_t^φ at any time t. This strategy is unique and called *risk-minimizing.*

Theorem 3.5.2: There exists a unique admissible risk-minimizing strategy $\varphi = (\vartheta, \eta)$ given by

$$(\vartheta_t, \eta_t) = (\vartheta_t^H, V_t^\star - \vartheta_t^H S_t^\star),\ 0 \le t \le T.$$

The risk process is given by $R_t^\varphi = E^\star\big[(L_T^H - L_t^H)^2|\mathcal{F}_t\big]$, and is called *the intrinsic risk process.*

It is interesting to remark that an admissible risk-minimizing strategy is mean-self-financing.

We have seen that the insurer's liability is given by

$$H_T = \int_0^T g(u, S_u) B_u^{-1} dN_u.$$

Following Moller [23] the intrinsic value process of H_T is given in this case by

$$V_t^\star = E^\star[H_T|\mathcal{F}_t] = \int_0^t g(u, S_u) B_u^{-1} dN_u + E^\star\left[\int_t^T g(u, S_u) B_u^{-1} dN_u|\mathcal{F}_t\right]$$

$$= \int_0^t g(u, S_u) B_u^{-1} dN_u + E^\star\left[\int_t^T g(u, S_u) B_u^{-1} (l_x - N_t)_{n-t}p_{x+t}\mu_{x+u} du|\mathcal{F}_t\right]$$

$$= \int_0^t g(u, S_u) B_u^{-1} dN_u + E^\star\left[\int_t^T g(u, S_u) B_t^{-1} e^{-\int_t^u r_\tau d\tau} (l_x - N_t)_{n-t}p_{x+t}\mu_{x+u} du|\mathcal{F}_t\right].$$

Let us define now

$$F^g(t, S_t) = E^\star[e^{-\int_t^u r_\tau d\tau} g(u, S_u)|\mathcal{F}^f] = E^\star[e^{-\int_t^u r_\tau d\tau} g(u, S_u)|\mathcal{F}]$$

which is the unique arbitrage-free price at time t of the claim g in the complete financial model with filtration $\mathcal{F}^f$. Using this function, the instrinsic process can be written as:

$$V_t^\star = \int_0^t g(u, S_u) B_u^{-1} dN_u + \int_t^T F^g(t, S_t) B_t^{-1} (l_x - N_t)_{n-t}p_{x+t}\mu_{x+u} du$$

Next, we are trying to find

$$d(B_t^{-1} F^g(t, S_t)) = -r(t, S_t) B_t^{-1} F^g(t, S_t) dt + B_t^{-1} dF^g(t, S_t)$$

$$= -r(t, S_t) B_t^{-1} F^g(t, S_t) dt + B_t^{-1} \left(F_t^g(t, S_t) dt + F_s^g(t, S_t) dS_t + \frac{1}{2} F_{ss}^g(t, S_t) \sigma(t, S_t)^2 S_t^2 dt\right)$$

$$= F_s^g(t, S_t) dS_t^\star.$$

as $dS_t = S_t^\star dB_t + B_t dS_t^\star = S_t^\star(r_t B_t dt) + B_t dS_t^\star = S_t r_t dt + B_t dS_t^\star$ and the price process $F^g(t, S_t)$ is characterized by the partial differential equation

$$-r(t,S_t)F^g(t,S_t) + F_t^g(t,S_t) + r(t,S_t)S_t F_s^g(t,S_t) + \frac{1}{2}\sigma(t,S_t)^2 S_t^2 F_{ss}^g(t,S_t) = 0,$$

with boundary condition $F^g(T, S_T) = g(T, S_T)$. Following Ikeda and Watanabe [18], the intrinsic process can be expressed as:

$$\begin{aligned} V_t^\star = V_0^\star &+ \int_0^t -F^g(\tau,S_\tau)B_\tau^{-1}(l_x - N_\tau)\mu_{x+r}d\tau \\ &+ \int_0^t \left(g(\tau,S_\tau)B_\tau^{-1} - \int_\tau^T B_\tau^{-1}F^g(\tau,S_\tau)_{u-\tau}p_{x+\tau}\mu_{x+u}du \right) dN_\tau \\ &+ \int_0^t \left(\int_\tau^T F^g(\tau,S_\tau)B_\tau^{-1}{}_{u-\tau}p_{x+\tau}\mu_{x+u}du \right)(l_x - N_{\tau^-})\mu_{x+\tau}d\tau \\ &+ \int_0^t \left((l_x - N_{\tau^-}) \int_\tau^T F^g(\tau,S_\tau)_{u-\tau}p_{x+u}\mu_{x+u}du \right) dS_\tau^\star. \end{aligned}$$

This gives the following decomposition of the intrinsic process

$$V_t^\star = V_0^\star + \int_0^t \vartheta_u^H dS_u^\star + \int_0^t \nu_u^H dM_u$$

where $M_t = N_t - \int_0^t \lambda_u du$ is the compensated counting process and

$$\vartheta_t^H = (l_x - N_{t^-}) \int_t^T F_s^g(t,S_t)_{u-t}p_{x+t}\mu_{x+u}du,$$

$$\nu_t^H = g(t,S_t)B_t^{-1} - \int_t^T F^g(t,S_t)B_t^{-1}{}_{u-t}p_{x+t}\mu_{x+u}du.$$

Combining this result with theorem 3.5.1. we get the following result:

Theorem 3.5.3. The unique admissible risk-minimizing strategy for the insurance company's contingent claim is given by:

$$\vartheta_t^\star = (l_x - N_{t^-}) \int_t^T F_s^g(t,S_t)_{u-t}p_{x+t}\mu_{x+u}du,$$

$$\eta_t^\star = \int_0^t g(u,S_u)B_u^{-1}dN_u + (l_x - N_t)\int_t^T F^g(t,S_t)B_t^{-1}{}_{u-t}p_{x+t}\mu_{x+u}du - \vartheta_t^\star S_t^\star,$$

for $0 \le t \le T$.

The value of the insurer's portfolio can be seen as the sum of benefits set aside for deaths already occured and the expectations of the benefits associated with future deaths:

$$V_t^\varphi = \int_0^t g(u,S_u)B_u^{-1}dN_u + E^\star\big[\int_t^T g(u,S_u)B_u^{-1}dN_u|\mathcal{F}_t\big].$$

It is interesting to note that, when a death occurs at time t, the reserves set by the insurance company for the benefits are relieved by the amount

$$\int_t^T F^g(t,S_t)B_t^{-1}{}_{u-t}p_{x+t}\mu_{x+u}du$$

Let us now consider a few different types of GMDB riders.

3.5.1 GMDB with return of premium

We start by analyzing a GMDB with return of premium (ROP). In this case, the function that models the contingent claim is given by $g(u,S_u)$ =max$(S_u, K = S_0) = K + (S_u - K)^+$. In this case (and assuming again that the financial market is complete), $F^g(t,S_t)$ can be evaluated by the Black-Scholes formula:

$$F^g(t,S_t) = E^\star\big[e^{-r(T-t)}(K + (S_T - K)^+)|\mathcal{F}_t\big]$$

$$= Ke^{-r(T-t)} + S_t\Phi(z_t) - Ke^{-r(T-t)}\Phi\big(z_t - \sigma\sqrt{T-t}\big)$$

$$= Ke^{-r(T-t)}\Phi\big(-z_t + \sigma\sqrt{T-t}\big) + S_t\Phi(z_t),$$

where Φ is the normal cumulative distribuition and

$$z_t = \frac{ln\big(\frac{S_t}{K}\big) + \big(r + \frac{\sigma^2}{2}\big)\big(T-t\big)}{\sigma\sqrt{T-t}}.$$

We also notice that the first order partial derivative with respect to s is $F^g_s(t, S_t) = \Phi(z_t)$ and hence, using theorem 3.5.2 we get the following hedging strategy:

$$\vartheta^\star_t = (l_x - N_{t^-}) \int_t^T \Phi(z_t)_{u-t}p_{x+t}\mu_{x+u} du,$$

$$\eta^\star_t = (l_x - N_t) \int_t^T \Big(K e^{-r(T-t)} \Phi\big(- z_t + \sigma\sqrt{T-t}\big) + S_t \Phi(z_t)\Big) B_t^{-1}{}_{u-t}p_{x+t}\mu_{x+u} du$$
$$+ \int_0^t g(u, S_u) B_u^{-1} dN_u - \vartheta^\star_t S^\star_t$$

3.5.2 GMDB with return of premium with interest

The next rider we can hedge is the return of premium with interest. In this case $g(u, S_u)$ =max$(S_u, Ke^{\delta u})$, where δ is the force of interest. Furthermore,

$$F^g(t, S_t) = K s^{\delta u} e^{-r(T-t)} \Phi\big(- z_t + \sigma\sqrt{u-t}\big) + S_t \Phi(z_t)$$

where now z_t is given by:

$$z_t = \frac{ln\big(\frac{S_t}{Ke^{\delta u}}\big) + \big(r + \frac{\sigma^2}{2}\big)\big(T - t\big)}{\sigma\sqrt{T-t}}.$$

Finally, using theorem 3.5.2. we get the risk-minimizing strategy:

$$\vartheta_t = (l_x - N_{t^-}) \int_t^T {}_{u-t}p_{x+t}\mu_{x+u} \Phi(z_t) du,$$

$$\eta_t = (l_x - N_t) \int_t^T {}_{u-t}p_{x+t}\mu_{x+u} K e^{-(r-\delta)u} \Phi(z_t + \sigma\sqrt{u-t}) du +$$
$$+ \int_0^t g(u, S_u) B_u^{-1} dN_u - \Delta N_t \int_t^T {}_{u-t}p_{x+t}\mu_{x+u} \Phi(z_t) S^\star_t du.$$

3.5.3 GMDB with ratchet

Recall, when a GMDB has a ratchet rider, there are some anniversary dates when the death benefit can be ratcheted up. The individual liability is $g(t, S_t)$ =max$(K =$

$S_0, S_1, S_2, \dots, S_t)$. We can apply the above strategy inductively, looking at periods bewtween two consecutive anniversary dates. So, let us consider $g_1(t, S_t) = \max(S_0, S_1) =: L_1$ and we apply the hedging strategy for the return of premium rider on the interval $[0, 1]$. Next function is $g_2(t, S_t) = \max(L_1, S_2) =: L_1$ and so we get the hedging strategy on $[1, 2]$. The trading strategy on $[0, 1]$ is given by:

$$\vartheta_t^\star = (l_x - N_{t^-}) \int_t^1 \Phi(z_t)_{u-t}p_{x+t}\mu_{x+u} du,$$

$$\eta_t^\star = (l_x - N_t) \int_t^1 \Big(Ke^{-r(1-t)}\Phi\big(-z_t + \sigma\sqrt{1-t}\big) + S_t\Phi(z_t)\Big) B_t^{-1}{}_{u-t}p_{x+t}\mu_{x+u} du$$
$$+ \int_0^t g_1(u, S_u) B_u^{-1} dN_u - \vartheta_t^\star S_t^\star$$

where

$$z_t = \frac{ln\big(\frac{S_t}{K}\big) + \big(r + \frac{\sigma^2}{2}\big)\big(1-t\big)}{\sigma\sqrt{1-t}}.$$

The hedging strategy is extended for $t > 1$ using similar formulas. Similar formulae can be found for other riders, using the same technique: roll-up, reset etc. It can also be shown that the ratio $\frac{\sqrt{R_0}}{l_x}$ converges to 0 as l_x increases, showing that this nonhedgeable part of the claim, actually its risk, decreases with the number of contracts sold. The variable annuity with a GMDB ratchet could be priced and hedged perfectly if instead of a single premium, the contract is sold for a variable (dynamic) premium process. Let's assume we have the following sample path for the stock price process given by Figure 3.3 and that the death benefit is payable at the end of the year of death. Let's assume that the death are uniformly distributed, and the maximum lifetime is ω. Let us assume the age of the insured is x. The present value of the benefits is given by the formula:

$$PV = \sum_{m=1}^{\omega-x} {}_{m-1|}q_x P(m, S_0) + \sum_{m=2}^{\omega-x} {}_{m-1|}q_x \big[P(m, S_1) - P(m, S_0)\big]_+ + \dots$$

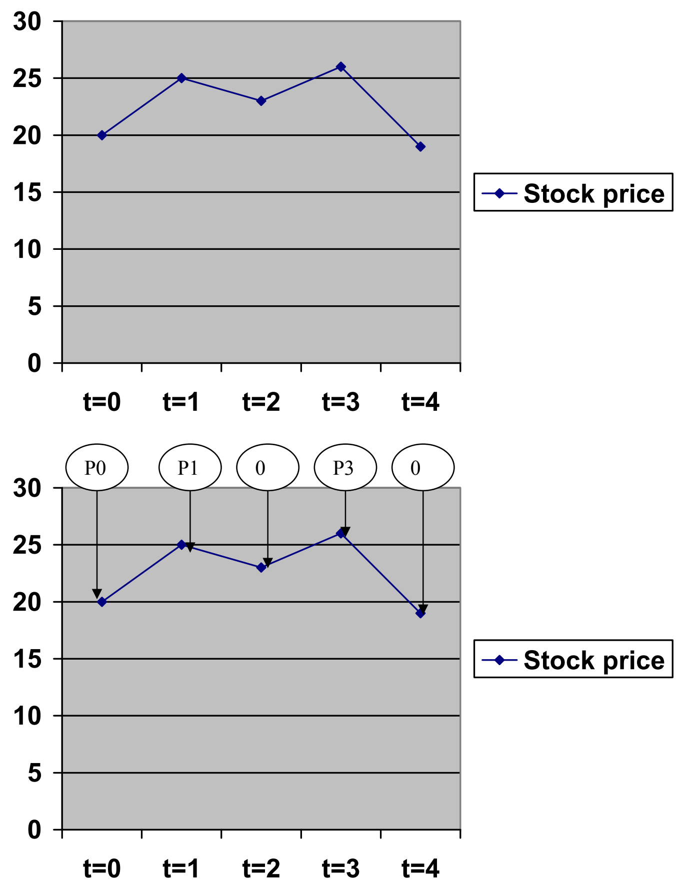

Figure 3.3: Sample stock price evolution

This allows us to sell this contract for a sequence of premiums. In our example (Figure 3.3) the first premium is

$$P_0 = \sum_{m=1}^{\omega-x} {}_{m-1|}q_x P(m, S_0).$$

At time 1, the stock price (value) is above P_0 and so the death benefit is ratcheted up and the insured has to pay (if he wants the death benefit ratcheted up) a new premium:

$$P_1 = \sum_{m=2}^{\omega-x} {}_{m-1|}q_x \big[P(m, S_1) - P(m, S_0)\big]_+.$$

At time 2, the stock price drops, and so this doesn't affect the death benefit. The premium at this time is 0. This process continues untill the expiration of the contract.

3.6 Living benefits

In the previous sections we have discussed benefits that are triggered by the death of the insured. We now take a look at benefits that are paid only if the person is alive at the end of the contract. These benefits are more expensive in the age group they are sold for (the probability of living untill the expiration of the contract is bigger than the probability of dieing). Their Canadian name is VAGLB (variable anuities guaranteed living benefits) while American insurance companies call them GMAB (guaranteed minimum accumulation benefits) or GMIB (guaranteed minimim income benefits - when the benefits are annuitized).

We consider a setting similar to the GMDB case. But, in this case, for living benefits, the present value of the claim is

$$X = g(S_T)B_T^{-1}(l_x - N_T),$$

and the intrinsic value process is given by

$$V_t^\star = E^\star[X|\mathcal{F}_t]$$

Following Moller [23] the stochastc independence between the lifetimes of the individuals and the market (i.e. between N and (B,S)) allows us to rewrite $V_t^\star$ as

$$\begin{aligned} V_t^\star &= E^\star\big[(l_x - N_T)|\mathcal{F}_t\big]B_t^{-1}E^\star\big[g(S_T)B_tB_T^{-1}|\mathcal{F}_t\big] \\ &= E^\star\Bigg[\sum_{i=1}^{l_x} I(T_i > T)|\mathcal{F}_t\Bigg]B_t^{-1}E^\star\big[g(S_T)B_tB_T^{-1}|\mathcal{F}_t\big] \\ &= \sum_{i:\ T_i>T} E^\star\big[I(T_i > T)|\mathcal{F}_t\big]B_t^{-1}E^\star\big[g(S_T)B_tB_T^{-1}|\mathcal{F}_t\big] \\ &= \sum_{i:\ T_i>T} {}_{T-t}p_{x+t}B_t^{-1}E^\star\big[g(S_T)B_tB_T^{-1}|\mathcal{F}_t\big] \end{aligned}$$

$$= (l_x - N_t)_{T-t}p_{x+t}B_t^{-1}E^\star\big[g(S_T)B_tB_T^{-1}|\mathcal{F}_t\big].$$

Also, note that the conditional dstribution of the market price processes doesn't depend on information about the insurance model ($\mathcal{F}_t^a$), and so

$$E^\star\big[g(S_T)B_tB_T^{-1}|\mathcal{F}_t\big] = E^\star\big[g(S_T)B_tB_T^{-1}|\mathcal{F}_t^f\big] = F^g(t, S_t).$$

Similar arguments to the GMDB case lead us to

$$V_t^\star = (l_x - N_t)_{T-t}p_{x+t}B_t^{-1}F^g(t, S_t).$$

It$\hat{o}$ formula is next applied giving us:

$$V_t^\star = V_0^\star + \int_0^t (l_x - N_{u^-})_{T-u}p_{x+u}\mu_{x+u}B_u^{-1}F^g(u, S_u)du$$

$$+ \int_0^t (l_x - N_{u^-})_{T-u}p_{x+u}d(B_u^{-1}F^g(u, S_u)) + \sum_{0<u\leq t}(V_u^\star - V_{u^-}^\star).$$

Recall from the GMDB case that

$$d(B_t^{-1}F^g(t, S_t)) = -r(t, S_t)B_t^{-1}F^g(t, S_t)dt + B_t^{-1}dF^g(t, S_t)$$

$$= -r(t, S_t)B_t^{-1}F^g(t, S_t)dt + B_t^{-1}\big(F_t^g(t, S_t)dt + F_t^g(t, S_t)dS_t + \frac{1}{2}F_{ss}^g(t, S_t)\sigma(t, S_t)^2S_t^2dt\big)$$

$$= F_s^g(t, S_t)dS_t^\star.$$

We also notice that

$$\sum_{0<u\leq t}(V_u^\star - V_{u^-}^\star) = -\int_0^t B_t^{-1}F^g(t, S_t)dt.$$

Hence, we can decompose the value process of the contingent claim X as

$$V_t^\star = V_0^\star + \int_0^t \vartheta_u^X dS_u^\star + \int_0^t \nu_u^X dM_u$$

where $M_t = N_t - \int_0^t \lambda_u du$ is the compensated counting process and

$$\vartheta_t^X = (l_x - N_{t^-})F_s^g(t, S_t)_{T-t}p_{x+t},$$

$$\nu_t^X = -B_t^{-1} F^g(t, S_t)_{T-t}p_{x+t}$$

for $0 \leq t \leq T$.

Therefore, we have the following:

Theorem 3.6.1: The adimissible strategies minimizing the variance $E^\star[(C_T^\varphi - E^\star C_T^\varphi)^2]$ are characterized by:

$$\vartheta_t^\star = (l_x - N_{t-}) F_s^g(t, S_t)_{T-t}p_{x+t},$$

$$\eta_T^\star = X - \vartheta_T S_T^\star.$$

The minimal variance can be determined [23] by use of Fubini's theorem:

$$E^\star\left[\left(\int_0^T \nu_u^X dM_u\right)^2\right] = E^\star\left[\int_0^T \left(\nu_u^X\right)^2 d\langle M\rangle_u\right]$$

$$= E^\star\left[\int_0^T \left(B_u^{-1} F^g(u, S_u)_{T-u}p_{x+u}\right)^2 \lambda_u du\right]$$

$$= \int_0^T E^\star\left[\left(B_u^{-1} F^g(u, S_u)\right)^2\right]_{T-u}p_{x+u}^2 E^\star[(l_x - N_u)\mu_{x+u}]du$$

$$= \int_0^T E^\star\left[\left(B_u^{-1} F^g(u, S_u)\right)^2\right]_{T-u}p_{x+u}^2 \; l_x \; {}_u p_x \mu_{x+u} du$$

$$= l_x \; {}_T p_x \int_0^T E^\star\left[\left(B_u^{-1} F^g(u, S_u)\right)^2\right]_{T-u}p_{x+u}\mu_{x+u} du.$$

Recall the Galtchouk-Kunita-Watanabe decomposition for the instrinsic value process:

$$V_t^\star = E^\star[X] + \int_0^t \vartheta_u^X dS_u^\star + L_t^X,$$

and we obtain

$$E^\star\left[(L_T^X - L_t^X)^2 | \mathcal{F}_t\right] = E^\star\left[\left(\int_t^T \nu_u^X dM_u\right)^2 | \mathcal{F}_t\right]$$

$$= E^\star\left[\int_t^T \left(\nu_u^X\right)^2 \lambda_u du | \mathcal{F}_t\right]$$

$$= \int_t^T E^\star\left[\left(\nu_u^X\right)^2 | \mathcal{F}_t\right] E^\star\left[(l_x - N_u)\mu_{x+u} | \mathcal{F}_t\right] du$$

$$= (l_x - N_u) \int_t^T E^\star \Big[\big(\nu_u^X\big)^2 | \mathcal{F}_t \Big]_{u-t} p_{x+t} \mu_{x+u} du.$$

We have now a theorem similar to 3.5.2. for living benefits:

Theorem 3.6.2: The unique admissible risk minimizing strategy for the living benefits is given by:

$$\vartheta_t^\star = (l_x - N_{t-})_{T-t} p_{x+t} F_s^g(t, S_t),$$

$$\eta_t^\star = (l_x - N_t)_{T-t} p_{x+t} F_s^g(t, S_t) B_t^{-1} - \vartheta_t^\star S_t^\star.$$

for $0 \le t \le T$.

The intrinsic risk process is given by:

$$(l_x - N_u) \int_t^T E^\star \Big[\big(\nu_u^X\big)^2 | \mathcal{F}_t \Big]_{u-t} p_{x+t} \mu_{x+u} du.$$

3.6.1 VAGLB with return of premium

In this case $g(u, S_u)$ =max$(S_u, K = S_0) = K + (S_u - K)^+$ and (assuming again that the financial market is complete), $F^g(t, S_t)$ can be evaluated by the Black-Scholes formula:

$$F^g(t, S_t) = E^\star \big[e^{-r(T-t)} (K + (S_T - K)^+) | \mathcal{F}_t \big]$$

$$= K e^{-r(T-t)} + S_t \Phi(z_t) - K e^{-r(T-t)} \Phi\big(z_t - \sigma\sqrt{T-t}\big)$$

$$= K e^{-r(T-t)} \Phi\big(- z_t + \sigma\sqrt{T-t}\big) + S_t \Phi(z_t),$$

where Φ is the normal cumulative distribuition and

$$z_t = \frac{ln\big(\frac{S_t}{K}\big) + \big(r + \frac{\sigma^2}{2}\big)\big(T-t\big)}{\sigma\sqrt{T-t}}.$$

We also notice that the first order partial derivative with respect to s is $F_s^g(t, S_t) = \Phi(z_t)$ and hence, using theorem 3.6.2 we get the following hedging strategy:

$$\vartheta_t = (l_x - N_{t-})_{T-t} p_{x+t} \Phi(z_t),$$

$$\eta_t = (l_x - N_t)_{T-t}p_{x+t}F^g(t, S_t)e^{-rt} - (l_x - N_t)_{T-t}p_{x+t}\Phi(z_t)S_t^\star.$$

$$= (l_x - N_t)_{T-t}p_{x+t}Ke^{-rt}\Phi\big(- z_t + \sigma\sqrt{T-t}\big) - \Delta N_{tT-t}p_{x+t}\Phi(z_t)S_t^\star,$$

while the instrinsic risk process is given by

$$R_t^\varphi = (l_x - N_t)_{T-t}p_{x+t}\int_t^T E^\star\big[(e^{-ru}F^g(u, S_u))^2|\mathcal{F}_t\big]_{T-u}p_{x+u}\mu_{x+u}du.$$

3.6.2 VAGLB with return of premium with interest

The next rider we can hedge for VAGLB is the return of premium with interest. In this case $g(u, S_u)$ =max$(S_u, Ke^{\delta u})$, where δ is the force of interest. Furthermore,

$$F^g(t, S_t) = Ks^{\delta u}e^{-r(T-t)}\Phi\big(- z_t + \sigma\sqrt{u-t}\big) + S_t\Phi(z_t)$$

where now z_t is given by:

$$z_t = \frac{ln\big(\frac{S_t}{Ke^{\delta u}}\big) + \big(r + \frac{\sigma^2}{2}\big)\big(T-t\big)}{\sigma\sqrt{T-t}}.$$

Finally we also notice that the first order partial derivative with respect to s is $F_s^g(t, S_t) = \Phi(z_t)$ and hence, using theorem 3.6.2. we get the risk-minimizing strategy:

$$\vartheta_t = (l_x - N_{t-})_{T-t}p_{x+t}\Phi(z_t),$$

$$\eta_t = (l_x - N_t)_{T-t}p_{x+t}F^g(t, S_t)e^{-rt} - (l_x - N_t)_{T-t}p_{x+t}\Phi(z_t)S_t^\star.$$

$$= (l_x - N_t)_{T-t}p_{x+t}Ks^{\delta u}e^{-r(T-t)}\Phi\big(-z_t + \sigma\sqrt{u-t}\big) + S_t\Phi(z_t)e^{-rt} - (l_x - N_t)_{T-t}p_{x+t}\Phi(z_t)S_t^\star.$$

3.7 Discrete time analysis

We consider a descrete time model, in which the financial market follows the Cox-Ross-Rubinstein model (also known as the binomial model). The model consists of two basic securities. The time horizon is T and the set of dates in the financial market model is $t = 0, 1, 2, \dots, T$. Assume that the first security is riskless (bond or bank account) B, with price process

$$B_t = (1+r)^t, \ t = 0, 1, \dots, T,$$

i.e. the bond yields a riskless rate of return r in each time interval $[t, t+1]$. The second security is a risky asset (stock, or stock index) S with price process

$$S(t+1) = \begin{cases} uS(t) & \text{with probability } p \\ dS(t) & \text{with probability } 1-p \end{cases}$$

for $t = 0, 1, \dots, T-1$ and with $0 < d < u$, and $S_0 \in \mathbf{R}_0^+$.

The first task is to find an equivalent martingale measure, i.e. a probability measure Q which is equivalent to the physical measure and such that the discounted price process $S^\star(t) := S(t)B^{-1}(t)$ is a martingale with respect to Q. In other words we want to determine q such that $Q(\{u\}) = q$ and $Q(\{d\}) = 1-q$ and Q satisfies the above conditions. We have the following result:

Theorem 3.7.1.

1. A martingale measure Q for the discounted stock price exists if and only if

$$d < 1 + r < u$$

2. If 1. holds true, then the measure Q is uniquely determined by:

$$q = \frac{1+r-d}{u-d}$$

For a proof, see [3]. This theorem tells us that the financial binomial market wich satisfies the natural condition 1. is complete. Hence, we have perfect hedging of options. It is interesting to notice that a so-called trinomial model (when the price process has 3 different outcomes) is not complete. The natural filtration in this model F^f is given by $\mathcal{F}_t^f := \sigma\{S_1, \ldots, S_t\}$. Let X be a contingent claim, that is a $\mathcal{F}_T^f$-measurable Q-integrable random variable. Define the following process $W_t = E^\star[X|\mathcal{F}_t^f]$ which is a martingale with respect to the filtration F^f and the measure Q. Then we have the following representation [30]:

$$W_t = W_0 + \sum_{j=1}^{t} \alpha_j \Delta S_j^\star,$$

where α_j is predictable (i.e. $\mathcal{F}_{j-1}^f$-measurable), for any $j = 1, 2, \ldots, t$. Now, we can think of W as the discounted value process under some strategy φ. So we have $V_T(\varphi) = X$ i.e. the terminal value of the strategy is the claim and

$$V_t(\varphi) = V_0(\varphi) + \sum_{j=1}^{t} \alpha_j \Delta S_j^\star.$$

Now the strategy $\varphi = (\vartheta, \eta)$ that replicates X is given by $\vartheta_t = \alpha_t$ while η is uniquely determined such that the strategy is also self-financing by

$$\eta_t = W_0 + \sum_{j=1}^{t} \vartheta_j \Delta S_j^\star - \vartheta_t S_t^\star.$$

As a consequence, the price of the contract should be the initial value of the self-financing strategy, which is $W_0 = E^\star[X]$.

Consider now the second model, with a portfolio of l_x policy-holders age x (at time 0) and let $l_x - N_t$ the number of survivors at time t. The lifetime of the individuals in the group are modelled by $T_1, \ldots, T_{l_x}$ which are independent and identically distributed random variables. Also, define ${}_tp_x := P(T_1 > t)$. Let us assume that the contract is modelled by the claim $g(S_T)$ and has present value

$$X = (l_x - N_t)g(S_T)B_T^{-1}$$

Hence we have a guaranteed living benefit contract. As in the continuous time model, we introduce a filtration on the actuarial model F^a given by

$$\mathcal{F}_t^a = \sigma\{N_1, \dots, N_t\}.$$

The filtration in the product space is defined by $\mathcal{F}_t = \mathcal{F}_t^f \vee \mathcal{F}_t^a = \sigma(\mathcal{F}_t^f \cup \mathcal{F}_t^a)$, the smallest σ-algebra containing both $\mathcal{F}_t^f$ and $\mathcal{F}_t^a$.

Following [24] we define two processes related to the financial market and the actuarial market respectively. In the financial market, recall that the unique price at time t of the contract with payment $g(S_T)$ at time T is given by

$$P_t^f := E^\star\big[g(S_T)B_T^{-1}|\mathcal{F}_t^f\big] = E^\star\big[g(S_T)B_T^{-1}\big] + \sum_{j=1}^{t} \alpha_j^f \Delta S_j^\star.$$

In the actuarial market, we introduce the process

$$M_t := E^\star\big[(l_x - N_T)|\mathcal{F}_t^a\big] = (l_x - N_t)_{T-t}p_{x+t},$$

the conditional expected number of survivors at time T.

The discounted value proces is

$$V_t^\star = E^\star\big[(l_x - N_T)g(S_T)B_T^{-1}|\mathcal{F}_t\big] = E^\star\big[g(S_T)B_T^{-1}|\mathcal{F}_t\big]E^\star\big[(l_x - N_T)|\mathcal{F}_t\big]$$

$$= E^\star\big[g(S_T)B_T^{-1}|\mathcal{F}_t^f\big]E^\star\big[(l_x - N_T)|\mathcal{F}_t^a\big] = P_t^f M_t.$$

and we obtain the recurrence:

$$V_t^\star - V_{t-1}^\star = P_t^f M_t - P_{t-1}^f M_{t-1} = \big(P_t^f - P_{t-1}^f\big)M_{t-1} + P_t^f(M_t - M_{t-1})$$

$$= M_{t-1}\alpha_t^f \Delta S_t^\star + P_t^f \Delta M_t,$$

which gives us the decomposition:

$$V_t^\star = V_0^\star + \sum_{j=1}^{t} M_{j-1}\alpha_j^f \Delta S_j^\star + \sum_{j=1}^{t} P_j^f \Delta M_j.$$

In this decomposition, $M_{j-1}\alpha_j^f$ is predictable and $\sum_{j=1}^{t} P_j^f \Delta M_j$ is a martingale. In can also be shown that $(\sum_{j=1}^{t} P_j^f \Delta M_j) S_t^\star$ is a martingale and hence, the unique risk-minimizing strategy [24] is given by:

$$\vartheta_t = (l_x - N_{t-1})_{T-(t-1)} p_{x+(t-1)} \alpha_t^f,$$

$$\eta_t = (l_x - N_t)_{T-t} p_{x+t} P_t^f - (l_x - N_{t-1})_{T-(t-1)} p_{x+(t-1)} \alpha_t^f S_t^\star.$$

The cost process for the strategy is given by

$$C_t(\varphi) = V_0^\star + \sum_{j=1}^{t} P_j^f \Delta M_j.$$

The risk that remains with the insurer who applies the risk-minimizing strategy can be assesed using the variance of the accumulated costs $C_T(\varphi)$:

$$Var^\star[C_T(\varphi)] = \sum_{t=1}^{T} E^\star\big[(P_t^f)^2 \Delta M_t^2\big] = \sum_{t=1}^{T} E^\star\big[(P_t^f)^2\big] E\big[\Delta M_t^2\big]$$

$$= \sum_{t=1}^{T} E^\star\big[(P_t^f)^2\big] E\big[Var\big[\Delta M_t|\mathcal{F}_{t-1}\big]\big] + Var\big[E\big[\Delta M_t|\mathcal{F}_{t-1}\big]\big]$$

$$= \sum_{t=1}^{T} E^\star\big[(P_t^f)^2\big] E\big[Var\big[{}_{T-t}p_{x+t}(l_x - N_t)|\mathcal{F}_{t-1}\big]\big]$$

$$= \sum_{t=1}^{T} E^\star\big[(P_t^f)^2\big]\ l_x\ {}_T p_x {}_{T-t}p_{x+t}\big(1 - {}_1p_{x+(t-1)}\big).$$

We will next look at an example (in discrete settings) and compare the risk-minimizing strategy with other hedging strategies discussed in 3.1.

3.7.1 Risk comparison

Let us start by looking at a numerical example. We consider a binomial tree model with 4 trading times, $t = 0, 1, 2, 3$. At time 0 the stock price is 100. At time 1, it can go up to 110 or down to 80. The entire sample process is described in the following Figure 3.4. For simplicity and without restricting the generality, we can assume that the time span between two consecutive trading times is 1 year. The market contains the stock and a bank account (bond) earning interest at the rate $r = .05$ or 5 percent per year. We will analyze a living benefit contract associated with the stock. We will assume that the start of the contract is $t = 0$ and the expiration is $t = 3$. We will also assume that we are dealing with a return of premium rider, i.e. the living benefit is the maximum between the account value at the expiration (account value = stock value) and the guaranteed return of premium. For simplicity we assume that the remaining lifetimes of the policy-holders are independent and exponentially distributed with hazard rate (force of mortality) μ. In this case, the survival probability is

$$ {}_tp_x = exp(-\mu t) $$

and so we obtain

$$ \begin{aligned} E\big[\Delta M_t^2\big] &= l_x \; {}_Tp_x {}_{T-t}p_{x+t}\big(1 - {}_1p_{x+(t-1)}\big) \\ &= e^{-2\mu T+\mu t}(1 - e^{-\mu}) \end{aligned} $$

This allows us to determine the variance of the accumulated cost associated with the risk-minimizing strategy:

$$ Var^\star[C_T(\varphi)] = l_x \sum_{t=1}^{T} E^\star\big[(P_t^f)^2\big]{}_Tp_x {}_{T-t}p_{x+t}\big(1 - {}_1p_{x+(t-1)}\big) $$

Recall $f(S_T)$=max(S_T, K), where $K = S_0$. To find the hedging strategy, we will have to find the call option price tree associated with the stock tree process (Figure 3.4).

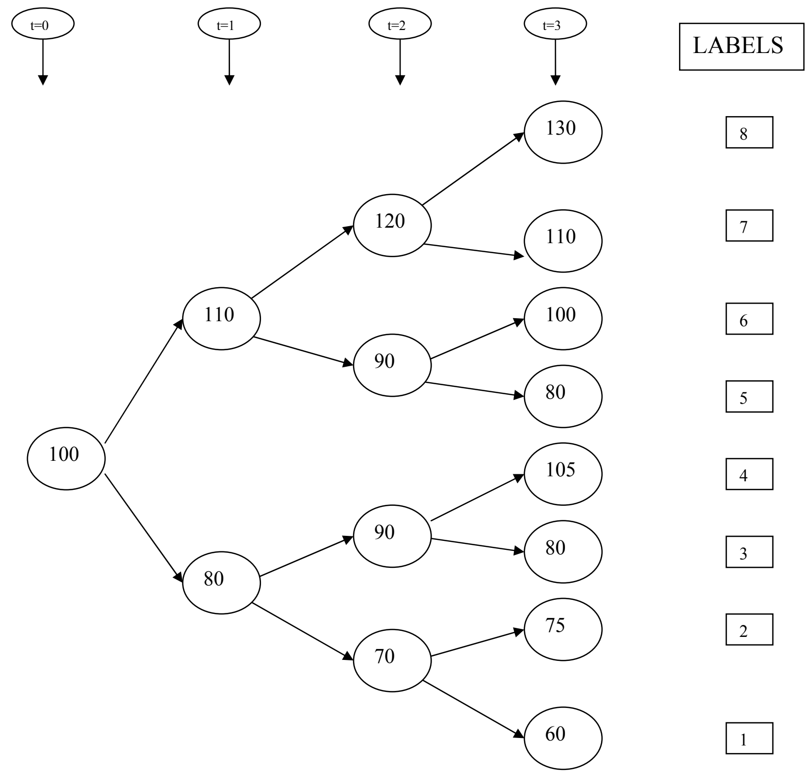

Figure 3.4: Binomial stock price process

We will find, at first, the risk-neutral probability for every branch of the tree. In other words, we want to find a measure $P^\star$ such that

$$S(t) = E^\star\left(\frac{S(t+1)}{1+r}\right), \ \forall\ t = 0, 1, 2$$

The first tree is $(100 - - > (110, 80))$ and we want to find $p^\star$ such that

$$100 = E^\star\left(\frac{S(1)}{1+r}\right) = \frac{110p^\star + 80(1-p^\star)}{1.05}.$$

Solving for $p^\star$ we get $p^\star = 5/6$. Using the same equation $S(t) = E^\star(\frac{S(t+1)}{1+r})$, we can find the risk-neutral probabilities for every brunch of the stock price tree and we can round up the tree in Figure 3.5. We will look at the financial probability space. Let $\Omega = \{1, 2, 3, 4, 5, 6, 7, 8\}$, the set of final states. We also define the σ-algebra $\mathcal{F} = \mathcal{P}(\Omega)$, the power set of Ω, and so $|\mathcal{F}| = 2^{|\Omega|}$. We also define a filtration $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \mathcal{F}_3$ where

$$\mathcal{F}_0 = \{\emptyset, \Omega\}$$

$$\mathcal{F}_1 = \sigma\{\{1, 2, 3, 4\}, \{5, 6, 7, 8\}\} = \{\emptyset, \{1, 2, 3, 4\}, \{5, 6, 7, 8\}, \Omega\}$$

$$\mathcal{F}_2 = \sigma\{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}\}$$

$\mathcal{F}_2$ has 2^4 elements and finally, $\mathcal{F}_3 = \mathcal{F}$. To define a probability measure on Ω, it is enough to define it on the set of simple events $\{\omega\}$, where $\omega \ \in \{\overline{1, 8}\}$. But for any state of the world $\{\omega\}$, there is a unique path from 0 to 3 and we will define the probability of the final state as the product of the conditional probabilities (given by the risk-neutral probabilities) along the path. Hence, we get:

$$P(\{8\}) = \frac{5}{6} \times \frac{17}{20} \times \frac{4}{5} = \frac{17}{30}$$

$$P(\{7\}) = \frac{5}{6} \times \frac{17}{20} \times \frac{1}{5} = \frac{17}{120}$$

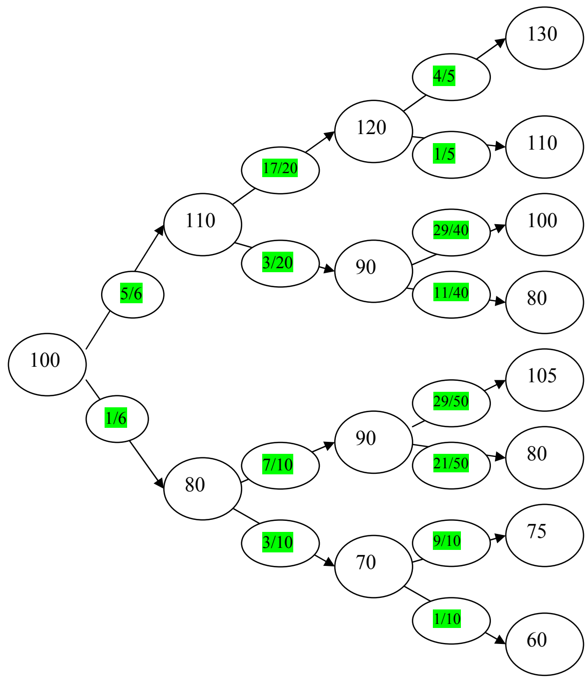

Figure 3.5: Risk-neutral probabilities

$$P(\{6\}) = \frac{5}{6} \times \frac{3}{20} \times \frac{29}{40} = \frac{29}{320}$$
$$P(\{5\}) = \frac{5}{6} \times \frac{3}{20} \times \frac{11}{40} = \frac{11}{320}$$
$$P(\{4\}) = \frac{1}{6} \times \frac{7}{10} \times \frac{29}{50} = \frac{203}{3000}$$
$$P(\{3\}) = \frac{1}{6} \times \frac{7}{10} \times \frac{21}{50} = \frac{49}{1000}$$
$$P(\{2\}) = \frac{1}{6} \times \frac{3}{10} \times \frac{9}{10} = \frac{9}{200}$$
$$P(\{1\}) = \frac{1}{6} \times \frac{3}{10} \times \frac{1}{10} = \frac{1}{200}.$$

Next we want to find the price of the call option $X(0)$ at time $t = 0$. The price is the discounted value of the expected payoff at $t = 3$:

$$X(0) = E^\star\left(\frac{\left(S(3) - K\right)_+}{1.05^3}\right) = \frac{40\left(\frac{17}{30}\right) + 20\left(\frac{17}{120}\right) + 10\left(\frac{29}{320}\right) + 15\left(\frac{203}{3000}\right)}{1.05^3} = 23.67$$

To determine the option price process, we will go backwords, using again the fact that the price at t is the discounted expected payoff. The tree is given by Figure 3.6. To determine the hedging strategy for the call option, we will use the option price process. Let $\varphi_0(1)$ the amount of bonds held at $t = 0$ and $\varphi_1(1)$ the number of shares of the stock at $t = 0$ that replicate the price of the call at $t = 0$ and its price at $t = 1$. We have the following system:

$$\begin{cases} 1.05\varphi_0 + 110\varphi_1 = 28.7 \\ 1.05\varphi_0 + 80\varphi_1 = 5.8 \end{cases}$$

Solving, we get $\varphi_1 = .76$ and $\varphi_0 = -52.68$ so we buy .76 shares of the stock and sell 52.68 in bonds. Continuing this process, we obtain the hedging process (i.e. number of shares process), given in Figure 3.6. Now, recall the formulas that gives us the hedging strategy for the living benefit:

$$\vartheta_t = (l_x - N_{t-1})_{T-(t-1)}p_{x+(t-1)}\alpha_t^f,$$

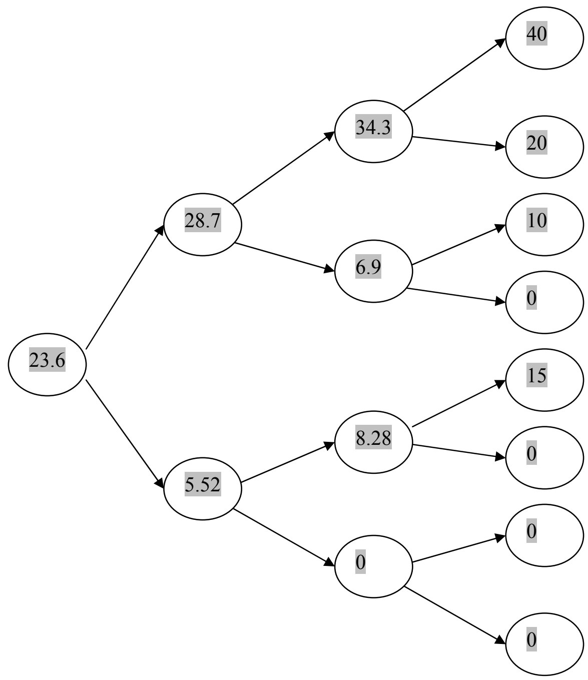

Figure 3.6: Call Option Price Process

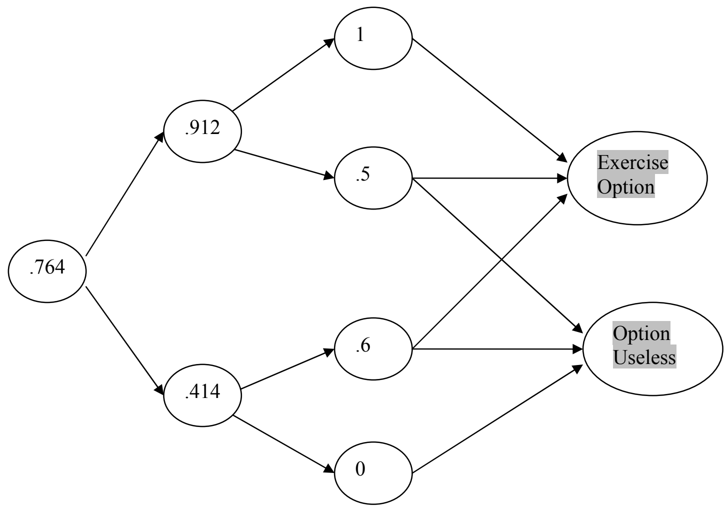

Figure 3.7: Hedging Process

$$\eta_t = (l_x - N_t)_{T-t}p_{x+t}P_t^f - (l_x - N_{t-1})_{T-(t-1)}p_{x+(t-1)}\alpha_t^f S_t^\star.$$

and the tree for the hedging process $\alpha_t^f = \varphi_1(t)$ (Figure 3.7). Let us assume that $\mu = 1$. Then, for one policy-holder we have

$$\vartheta_1 = {}_{T-(1-1)}p_{x+(1-1)}\alpha_1^f = e^{-3}(.764) = .038$$

and

$$\eta_1 = {}_{T-1}p_{x+1}P_1^f - {}_{T-(1-1)}p_{x+(1-1)}\alpha_1^f S_1^\star = e^{-2} \times 123.6 - .038 \times 100 = 12.92.$$

These are the number of shares (or bonds, respectively), held in the portfolio during the period $(0, 1]$. At time 1, these numbers will change as functions of the stock price evolution and the lifetime of the insured.

If the insured is not alive at $t = 1$, then $\vartheta_2 = 0$ and $\eta_2 = 0$.

If the insured is alive, then we have two cases. If the stock price is up (110), then

$$\vartheta_2 = {}_{T-(2-1)}p_{x+(2-1)}\alpha_2^f = e^{-2}(.912) = .123$$

and

$$\eta_2 = {}_{T-2}p_{x+2}P_2^f - {}_{T-(2-1)}p_{x+(2-1)}\alpha_2^f S_2^\star = e^{-1} \times 128.7 - .123 \times 104.76 = 34.46.$$

If the stock price is down (80), then

$$\vartheta_2 = {}_{T-(2-1)}p_{x+(2-1)}\alpha_2^f = e^{-2}(.414) = .056$$

and

$$\eta_2 = {}_{T-2}p_{x+2}P_2^f - {}_{T-(2-1)}p_{x+(2-1)}\alpha_2^f S_2^\star = e^{-1} \times 105.5 - .056 \times 104.76 = 32.94.$$

Continuing this algorithm, we come up with the risk-minimizing trading strategy, see Figure 3.8. If instead of $\mu = 1$ we take $\mu = 0.5$ we get a different risk-minimizing

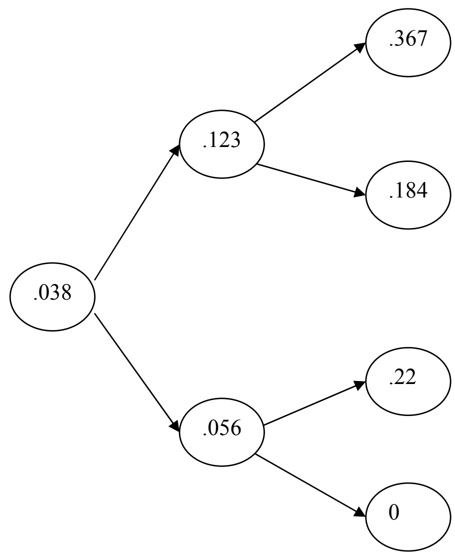

Figure 3.8: Risk-minimizing trading strategy, $\mu = 1$

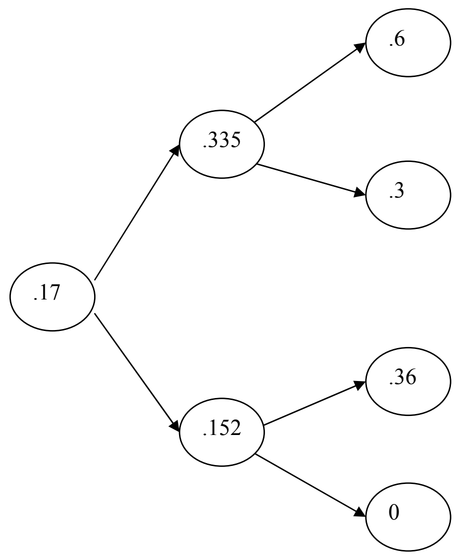

Figure 3.9: Risk-minimizing trading strategy, $\mu = 0.5$

trading strategy (see Figure 3.9.) Next we want to determine the variance of the accumulated cost $C_T(\varphi)$ for the two cases ($\mu = 1$, and $\mu = 0.5$). We have the following:

$$Var^\star[C_T(\varphi)] = l_x \sum_{k=1}^{T} E^\star\left[(P_k^f)^2\right]{}_T p_{xT-k} p_{x+k}\left(1 - {}_1p_{x+(k-1)}\right)$$

$$= \sum_{k=1}^{T} E^\star\left[(P_k^f)^2\right] e^{-2\mu T + \mu k}(1 - e^{-\mu})$$

and

$$Var^\star[X] = l_x E^\star\left[(P_T^f)^2\right] e^{-\mu T}(1 - e^{-\mu}) + l_x^2 Var^\star[P_T^f] e^{-2\mu T}$$

For $\mu = 1$ and $l_x = 1$ we get $Var^\star[C_T(\varphi)] = 736.92$ and $Var^\star[X] = 739.66$. The quotient $\frac{Var^\star[C_T(\varphi)]}{Var^\star[X]}$ is 0.996.

For $\mu = .5$ and $l_x = 1$ we get $Var^\star[C_T(\varphi)] = 2695.77$ and $Var^\star[X] = 2746.9$. The quotient $\frac{Var^\star[C_T(\varphi)]}{Var^\star[X]}$ is 0.981.

Generally, this quotient is an increasing function of μ because

$$\frac{Var^\star[C_T(\varphi)]}{Var^\star[X]} = \frac{1 - e^{-\mu}}{\left(1 - \frac{Var^\star[P_1^f]}{E^\star[(P_1^f)^2]}\right)(1 - e^{-\mu}) + \frac{Var^\star[P_1^f]}{E^\star[(P_1^f)^2]}}.$$

Finally, let us review the pricing techniques discussed above (Table 3.1).

Method	Price
Risk-minimizing	$l_{xT}p_x \times V_0^f$
Mean-variance hedging	$l_{xT}p_x \times V_0^f$
Super-hedging	$l_x \times V_0^f$

Table 3.1: Pricing formulas for living benefits contracts

3.8 Multiple decrements for variable annuities

Let us consider now a model with two random variables: T is the time until termination from a status and J, the cause of decrement. For simplicity, assume that J is a discrete random variable, and $|J| = m$. The joint probability density function of T and J is $f_{T,J}(t,j)$. We have the following standard relations and definitions. The probability of decrement due to cause j before time t is

$$ {}_tq_x^{(j)} = Pr[(0 < T \leq t) \cap (J = j)] = \int_0^t f_{T,J}(s,j)ds $$

Also, τ means all causes and we have

$$ {}_tq_x^{(\tau)} = Pr(T \leq t) = F_T(t) = \sum_{j=1}^{m} {}_tq_x^{(j)} $$

which is the probability of termination due to any cause,

$$ {}_tp_x^{(\tau)} = Pr(T > t) = 1 - {}_tq_x^{(\tau)} $$

which is the probability of survival with respect to all decrements and

$$ \mu_x^{(\tau)}(t) = -\frac{d}{dt} log\left({}_tp_x^{(\tau)}\right) = \sum_{j=1}^{m} \mu_x^{(j)}(t) $$

is the force of mortality to all causes.

The basic type of variable annuity with multiple decrements riders is defined for only two decrements, let us say death and invalidity. Once a decrement kicks in, the benefit is locked and paid at the expiration of the contract (ω). So, only one decrement can occur for each policyholder. The benefit is

$$ \begin{cases} H_1(T) & \text{if } T \leq \omega \text{ and } J = 1 \\ H_2(T) & \text{if } T \leq \omega \text{ and } J = 2 \\ S(\omega) & \text{if } T > \omega \end{cases} $$

where H_1 and H_2 are two functions that define the benefit for each decrement. As in the single decrement case,

$$N_t = \sum_{i=1}^{l_x} I(T \leq t)$$

One can also define

$$N_{jt} = \sum_{i=1}^{l_x} I((T \leq t) \cap (J = j))$$

for any $j = 1, ..., m$. The insurer's liability is now a function of the decrement also. So, the present value of the benefit is:

$$H_T = \int_0^T g(u, S_u) B_u^{-1} dN_u.$$

where $g(t, S_t)$ is the individual liability (depends on the decrement).

The intrinsic value process is given by

$$V_t^\star = E^\star[H_T | \mathcal{F}_t] =$$

$$= E^\star \left[\sum_{j=1}^{m} \int_0^T g_j(u, S_u) B_u^{-1} (l_x - N_u)_u p_x^{(\tau)} \mu_{x+u}^{(j)} du \mid \mathcal{F}_t \right]$$

Similiar computations to the single decrement case lead us to the following result:

Theorem 3.8.1. The unique admissible risk-minimizing strategy for the insurance company's contingent claim is given by:

$$\vartheta_t^\star = (l_x - N_{t^-}) \sum_{j=1}^{m} \int_t^T F_s^{(g_j)}(t, S_t)_{u-t} p_{x+t}^{(\tau)} \mu_{x+u}^{(j)} du,$$

$$\eta_t^\star = \int_0^t g_j(u, S_u) B_u^{-1} dN_{ju} + (l_x - N_t) \sum_{j=1}^{m} \int_t^T F^{g_j}(t, S_t) B_t^{-1}{}_{u-t} p_{x+t}^{(\tau)} \mu_{x+u}^{(j)} du - \vartheta_t^\star S_t^\star,$$

for $0 \leq t \leq T$.

A more complicated type of variable annuity with multiple decrements riders can be defined if the benefit is not paid at the time of the decrement occurance, but it is just locked; if another decrement occurs after that but before the expiration of the contract and if the benefit is larger that what was already locked, the policyholder will lock the larger benefit instead (which is then paid at the expiration).

The risk-minimizing technique can also be adapted in this case, although the formulas are a lot more complicated that in the multiple decrement case when the benefit is paid at the moment of the decrement occurance.

CHAPTER 4

RESULTS AND CONCLUDING REMARKS

In this dissertation, I am mostly interested in bringing a more theoretical approach into the problem of pricing and hedging variable annuities. There wasn't much research of this type in this area until recently. Most insurance companies use simulations in pricing these contracts, and there is a need for better techniques because of the size of the market (which exceeds one trillion dollars, accoring to Moody's) and the increased volatility of the stock market. This need is even larger in the case of incomplete markets (with no perfect hedging of options). We considered the problem of pricing and hedging for the most common riders attached to variable annuities and we also looked at risk-minimizing strategies and at a possible approach for discrete models.

There are two ways one can define an incomplete market.

We are dealing with a dual market model, financial and actuarial. If the financial model is incomplete, then the product market is also incomplete. We use an incomplete financial market model, in which the stock prices jump in different proportions at some random times which correspond to the jump times of a Poisson process. Between the jumps the risky assets follow the Black-Scholes model.

The product space is incomplete even if the financial market model is complete, because the actuarial risk (mortality risk) is not hedgeable in the stock market. The approach in this case follows the risk minimization technique defined by Follmer and Sondermann [12] and the work of Moller [23].

Both these alternatives were analyzed and reviewed.

APPENDIX A

KUNITA-WATANABE DECOMPOSITION

One of the most important results used in pricing and hedging in incomplete markets is the fundamental decomposition result of Kunita and Watanabe:

Theorem A1 For every $M \in \mathcal{M}_2$, we have the decomposition $M = N + Z$, where $N \in \mathcal{M}_2^{\star}$, $Z \in \mathcal{M}_2$, and Z is orthogonal to every element of $\mathcal{M}_2^{\star}$.

The proof follows [19]. Here $\mathcal{M}_2$ represents the space of square-integrable martingales and $\mathcal{L}^{\star}$ denotes the set of progressively measurable processes. Also, $\mathcal{M}_2^{\star}$ is the subset of $\mathcal{M}_2$ which consists of continuous stochastic integrals

$$I_t(X) = \int_0^t X_s dW_s;\ 0 \le t < \infty$$

where $X \in \mathcal{L}^{\star}$ and W is a Brownian motion. We have to show the existance of a process $Y \in \mathcal{L}^{\star}$ such that $M = I(Y) + Z$, where $Z \in \mathcal{M}_2$ has the property

$$\langle Z, I(X) \rangle = 0;\ \forall\ X \in \mathcal{L}^{\star} \tag{A.1}$$

Such a decomposition is unique (up to indistinguishability): if $M = I(Y^1) + Z^1 = M = I(Y^2) + Z^2$, with $Y^1, Y^2 \in \mathcal{L}^{\star}$ and both Z^1 and Z^2 satisfy (A1), then

$$Z := Z^2 - Z^1 = I(Y^1 - Y^2)$$

is a continuous element of $\mathcal{M}_2$ and $\langle Z \rangle = \langle Z, I(Y^1 - Y^2) \rangle = 0$. This implies that $P[Z_t = 0, \ \forall\ 0 \leq t < \infty] = 1$.

Therefore, it is enough to establish the decomposition for every finite time interval $[0, t]$; by uniqueness we can extend it to the entire half-line $[0, \infty)$. Fix $T > 0$ and let $\mathcal{R}_T$ be the closed subspace of $L^2(\Omega, \mathcal{F}_T, P)$ defined by

$$\mathcal{R}_T = \{I_T(X); \ X \in \mathcal{L}_T^{\star}\}$$

and let $\mathcal{R}_T^{\perp}$denote its orthogonal complement. $\mathcal{L}_T^{\star}$ denotes the class of processes $X \in \mathcal{L}^{\star}$ for which $X_t(\omega) = 0, \ \forall t > T, \ \omega \in \Omega$. Then the random variable M_T is in $L^2(\Omega, \mathcal{F}_T, P)$ and so it admits the decomposition

$$M_T = I_T(Y) + Z_T, \tag{A.2}$$

where $Y \in \mathcal{L}_T^{\star}$ and $Z_T \in L^2(\Omega, \mathcal{F}_T, P)$ satisfies

$$E[Z_T I_T(X)] = 0; \ \forall X \in \mathcal{L}_T^{\star}. \tag{A.3}$$

Let us denote by Z a right-continuous version of the martingale $E(Z_T|\mathcal{F}_t)$. Note that $Z_t = Z_T, \ \forall t \geq T$. We have $Z \in \mathcal{M}_2$ and conditioning (A2) on $\mathcal{F}_t$ we get

$$M_t = I_t(Y) + Z_t; \ 0 \leq t \leq T, \ a.s. \ P \tag{A.4}$$

Now it only remains to show that Z is orthogonal to every square-integrable martingale of the form $I(X); \ X \in \mathcal{L}_T^{\star}$. This is equivalent to showing that $\{Z_t I_t(X), \ 0 \leq t \leq T\}$ is a $\mathcal{F}_t$-martingale. This is true if $E[Z_S I_S(X)] = 0$ for every stopping time S of the filtration $\{\mathcal{F}_t\}$ with $S \leq T$. Finally,

$$E[Z_S I_S(X) = E[E(Z_T|\mathcal{F}_S) I_S(X)] = E[Z_T I_T(X_t(\omega) 1_{\{t \leq S(\omega)\}})] = 0$$

which proves the theorem.

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