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Viability : Models, algorithm and Applications in Finance and Environmental-Economics

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Warning: These notes are extracted from the forthcoming book **Viability, Control and Games: Regulation of Complex Evolutionary Systems under Uncertainty and Viability Constraints** by Jean-Pierre Aubin, Alexandre Bayen, Noël Bonneuil and Patrick Saint-Pierre to appear at Springer-Verlag in 2008.

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0.1.1 Chance and Necessity

It is by now a consensus that the evolution of many variables describing systems, organizations, networks arising in biology and human and social sciences do not evolve in a deterministic way, and in many instances, not even in a stochastic way as it is usually understood, but with a Darwinian flavor.

Viability theory started in 1976 by translating mathematically the title

$$\boxed{\begin{array}{ccc} \textit{Chance} & \textit{and} & \textit{Necessity} \\ \Updownarrow & & \Updownarrow \\ x'(t) \in F(x(t)) & \& & x(t) \in K \end{array}}$$

of the famous 1973 book by Jacques Monod, taken from an (apocryphical?) quotation of Democritus who held that *"the whole universe is but the fruit of two qualities, chance and necessity"*.

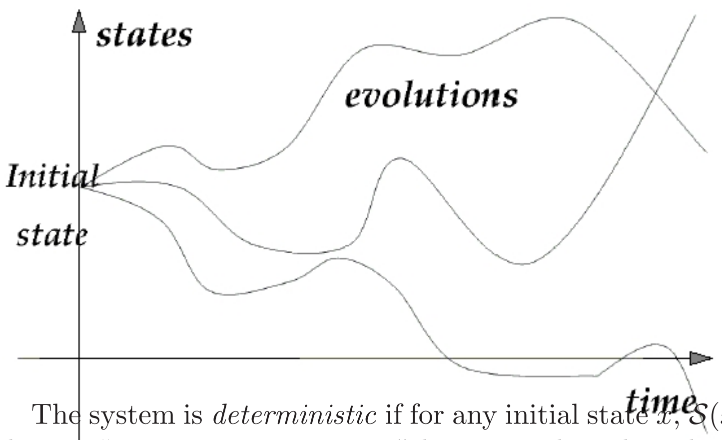


Figure 1: [**The mathematical translation of "chance".**]

*The mathematical translation of "**chance**" is the differential inclusion $x'(t) \in F(x(t))$, which is a kind of evolutionary engine (called an* evolutionary system*) associating with any initial state x the subset $\mathcal{S}(x)$ of evolutions starting at x and governed by the differential inclusion. The scheme above displays evolutions starting from a give initial states, which are functions from time (in abscissas) to the state space (ordinates).*

The system is *deterministic* if for any initial state x, $\mathcal{S}(x)$ is made of one and only one evolution, whereas "contingent uncertainty" happens when the subset $\mathcal{S}(x)$ of evolutions contains more than one evolution for at least one initial state. "Contingence is a non-necessity, it is a characteristic attribute of freedom", wrote *Gottfried Leibniz*.

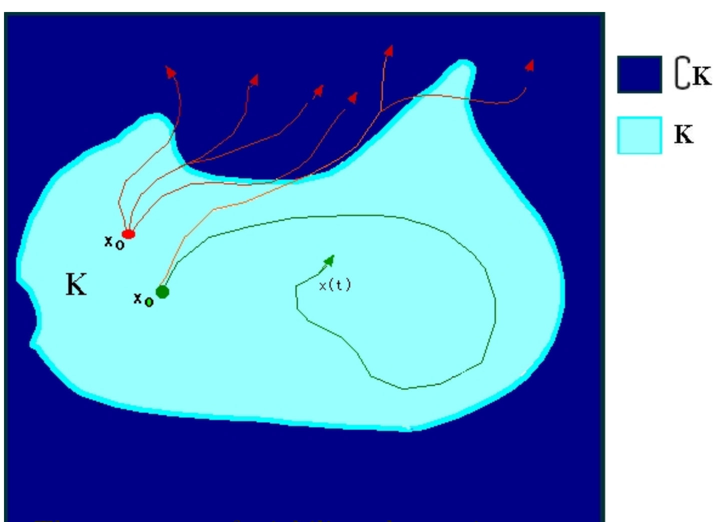


Figure 2: [**The mathematical translation of "necessity"**]

*The mathematical translation of "**necessity**" is the requirement that for all $t \geq 0$, $x(t) \in K$, meaning that at each instant, "viability constraints" are satisfied by the state of the system. The scheme above represents the state spaces as the plane, and the environment defined a subset. It shows two initial sates, one from which all evolutions violate the constraints in finite time, the other one from which starts one viable evolution and another one which is not viable.*

The purpose of viability theory is to attempt to answer directly the question that some economists, biologists or engineers ask: *"Complex organizations, systems and networks, yes, but for what purpose?"* The answer we suggest: *"to adapt to the environment."*

This is the case in economics when we have to adapt to scarcity constraints, balances between supply and demand, and many other constraints.

This is also the case in biology, since Claude Bernard's "constance du milieu intérieur" and Walter Cannon's "homeostasis". This is naturally the case in ecology and environmental studies.

This is equally the case in automatics and, in particular, in robotics, when the state of the system must evolve while avoiding obstacles forever or until they reach a target.

In summary, *the environment is described by viability constraints* of various kinds, a word encompassing polysemous concepts as *stability, confinement, homeostasis, adaptation*, etc., expressing the idea that some variables must obey some constraints (representing physical, social, biological and economic constraints, etc.) that can never be violated. So, viability theory started as *the confrontation of evolutionary systems governing evolutions and viability constraints* that such evolutions must obey.

In the same time, controls, subsets of controls, in engineering, regulons (regulatory controls) such as prices, messages, coalitions of actors, connectionist operators in biological and social sciences, which parameterize evolutionary systems, do evolve: *Their evolution must be consistent with the constraints, and the targets or objectives they must reach in finite or prescribed time.* The aim of viability theory is to provide the "regulation maps" associating with any state the (possibly empty) subset of controls or regulons governing viable evolutions.

Together with the selection of evolutions governed by teleological objectives, mathematically translated by intertemporal optimality criteria as in optimal control, viability theory offers other selection mechanisms by requiring evolutions to obey several forms of "*viability* requirements".

In social and biological sciences, intertemporal optimization can be replaced by *myopic, opportunistic, conservative and lazy* selection mechanisms of viable evolutions that involve present knowledge, sometimes the knowledge of the history (or the path) of the evolution, instead of anticipations or knowledge of the future (whenever the evolution of these systems cannot be reproduced experimentally). Other forms of uncertainty do not obey statistical laws, but takes also into account unforeseeable rare events (tyches, or perturbations, disturbances) that that must be avoided at all costs (precaution principle). These systems can be regulated by using regulation controls that have to be chosen as feedbacks for guaranteeing the viability of constraints and/or the capturability of targets and objectives, possibly against perturbations played by "Nature", called *tyches*.

However, there is no reason why collective constraints are satisfied at each instant by evolutions under uncertainty governed by evolutionary systems. This leads us to the study of *how to correct either the dynamics, and/or the constraints* in order to reestablish viability. This may allow us to provide an explanation of the formation and the evolution of controls and regulons through regulation or adjustment laws that can be designed (and computed) to insure viability, as well as other procedures, such as using *impulses* (evolutions with infinite velocity) governed by other systems, or by regulating the evolution of the environment.

Presented in such an evolutionary perspective, this approach of (complex) evolutionary systems departs from the main stream of modelling evolution by a a direct approach:

Direct Approach. It consists in studying properties of evolutions governed by an evolutionary system: Gather the larger number of properties of evolutions starting from each initial state. It may be an information both costly and useless, since our brains cannot handle simultaneously too many observations and concepts.

Moreover, it may happen that

1. evolutions starting from a given initial state satisfy properties which are lost by evolutions starting from another initial state, even close to it (sensitivity analysis),

2. or that, even if all evolutions share a given set of properties, they fade away for neighboring systems (stability analysis).

Viability theory rather uses instead an *inverse approach*:

Inverse Approach. A set of prescribed properties of evolutions being given, study the (possibly empty) subsets of initial states from which

1. starts at least one evolution governed by the evolutionary system satisfying the prescribed properties,

2. all evolutions starting from it satisfy these prescribed properties.

These two subsets coincide whenever the evolutionary system is deterministic.

Stationarity, periodicity and asymptotic behavior are examples of classical properties motivated by physical sciences which have been extensively studied. We shall add to this list the *viability* of an environment and the *capturability* of a target in finite time, and other problems of combining properties of this kind:

Definition 0.1.1. [**Viability and Capturability**] *If a subset* $K \subset \mathbb{R}^d$ *is regarded as an* environment *(defined by* viability constraints*), an evolution* $x(\cdot)$ *is said to be* viable in the environment $K \subset \mathbb{R}^d$ on an interval $[0, T[$ *(where* $T \leq +\infty$*) if for every time* $t \in [0, T[$, $x(t)$ *belongs to* K.
If a subset $C \subset K$ *is regarded as a target, an evolution* $x(\cdot)$ captures C *if there exists a finite time* T *such that the evolution is viable in* K *on the interval* $[0, T[$ *until it reaches the target at* $x(T) \in C$ *at time* T. *See Definition 1.1.3, p.16.*

0.1.2 Motivating applications

For dealing with these issues, one needs *"dedicated"* concepts and formal tools, algorithms and mathematical techniques motivated by complex systems evolving under uncertainty. For instance, and without entering into the details, we can mention systems sharing such common features arising in

1. **Systems designed by human brains** in the sense that agents, actors, decision-makers act on the evolutionary system, as in engineering. **Control theory and differential games,** conveniently revisited, provide many metaphors and tools for grasping

viability questions. Problems in *control design, stability, reachability, intertemporal optimality, tracking of evolutions, observability, identification and set-valued estimation*, etc., can be formulated in terms of viability and capturability concepts investigated in these lecture notes.

Some technological systems (robot of all kinds, from drones, underwater vehicles, etc., to animats) need "embedded systems" *autonomous* enough to regulate viability/capturability problems by adequate regulation (feedback) control laws.

2. **Systems observed by human brains**, more difficult to understand since human beings did not design or construct them. Human beings live, think, are involved in socio-economic interactions, but struggle for grasping why and how they do it, at least, why. This happens for instance in

   - **economics,** where the viability constraints are the scarcity constraints among many other ones. We can replace the fundamental Walrasian model of resource allocations by decentralized dynamical model in which the role of the controls is played by the prices or other economic decentralizing messages (as well as coalitions of consumers, interest rates, and so forth). The regulation law can be interpreted as the behavior of Adam Smith's invisible hand choosing the prices as a function of allocations of commodities,
   - **finance,** where shares of assets of a portfolio play the role of controls for guaranteing that the values of the portfolio remains above a given time/price dependent function at each instant until the exercise time (horizon), whatever the prices and their growth rates taken between evolving bounds,
   - **dynamical connectionist networks and/or dynamical cooperative games,** where coalitions of player may play the role of controls: each coalition acts on the environment by changing it through dynamical systems. The viability constraints are given by the architecture of the network allowed to evolve,
   - **genetics and population genetics,** where the viability constraints are the ecological constraints, the state describes the phenotype and the controls are genotypes or fitness matrices.
   - **sociological sciences,** where a society can be interpreted as a set of individuals subjected to viability constraints. They correspond to what is necessary for the survival of the social organization. Laws and other cultural codes are then devised to provide each individual with psychological and economical means of survival as well as guidelines for avoiding conflicts. Subsets of cultural codes (regarded as cultures) play the role of regulation parameters.

- **cognitive sciences,** where, at least at one level of investigation, the variables describe the sensory-motor activities of the cognitive system, while the controls translate into what could be called conceptual controls (which are the synaptic matrices in neural networks.)

Theoretical results about the above ways to think long term viability are useful for the understanding of non teleological evolutions, of inertia principle, of emergence of new regulons when viability is at stakes, of the role of different kinds of uncertainties (contingent, tychastic or stochastic), the (re)designing of regulatory institutions (regulated markets when political convention must exist for global purpose, mediation or metamediation of all kinds, including law, social conflicts, institutions for sustainable development, etc.) And progressively, when more data gathered by these institutions will be available, qualitative (and sometimes quantitative) prescriptions of viability theory may be useful.

0.2 Some Problems and Concepts of Viability Theory

The viability tools presented in these lecture notes are meant to enrich the panoply of those diverse and ingenious techniques set out by the study of dynamical systems since the pioneering works of Lyapunov and Poincaré more than one century ago. Most of them were motivated by physics and mechanics, not necessarily designed to adaptation problems to environmental or viability constraints. Viability theory incorporates some mathematical features of uncertainty without statistical regularity, deals not only with optimality but also with viability and *decisions taken at the appropriate time*. Viability techniques are also *geometric* in nature, but they do not require smoothness properties usually assumed in differential geometry. They not only deal with asymptotic behavior, but also and mainly with *transient* evolutions and capturability of targets in finite or prescribed time. They are *global* instead of local, and really *nonlinear* since they bypass linearization techniques of the dynamics around equilibria, for instance. They bring other lights to the decipherability of complex, paradoxical and strange dynamical behaviors by providing other types of mathematical results and algorithms. And above all, they have been motivated by dynamical systems arising in issues involving living beings, as well as networks of systems (or organizations, organisms).

In a nutshell, viability theory investigates evolutions

1. in *continuous time, discrete time, or an "hybrid"* of the two when *impulses* are involved,
2. constrained to *adapt* to an environment,

3. evolving under *contingent, stochastic or tychastic uncertainty*
4. using for this purpose *controls*, *regulons* (regulation controls), subsets of regulons, and in the case of networks, connectionist matrices,
5. *regulated* by *feedback laws* (static or dynamic) that are then ”computed” according to given principles, such as the *inertia principle*, intertemporal optimization, etc.,
6. co-evolving with their environment (*mutational and morphological viability*),
7. and corrected by introducing adequate controls (*viability multipliers*) when viability or capturability is at stakes.

0.3 Restoring Viability

There is no reason why an arbitrary subset K should be viable under a control system. One can imagine several other methods for this purpose:

1. Keep the constraints and change initial dynamics by introducing regulons that are “viability multipliers”.
2. Keep the same dynamics and
   - replace the initial environment by its viability kernel, i.e., the subset of initial states from which starts at least one viable solution,
   - let the set of constraints evolve according to *mutational equations*,
3. or change the initial conditions by introducing a *reset map* Φ mapping any state of K to a (possibly empty) set $\Phi(x) \subset X$ of new “initialized states” (*impulse control*).

Chapter 1

Viability and Capturability

1.1 Evolutions

Let X denote the state space of the system. Evolutions describe the behavior of the state of the system as a function of the time.

Definition 1.1.1. [***Evolutions and their Trajectories***] *The time t ranges over a set $\mathbb{T}$ that is in most cases,*

1. *either the **discrete time set** of times $j \in \mathbb{T} := \mathbb{N} := \{0, \ldots, +\infty\{$ ranging over the set of nonnegative integers $j \in \mathbb{N}$,*
2. *or the **continuous time set** of times $t \in \mathbb{T} := \mathbb{R}_+ := [0, \ldots, +\infty[$ ranging over the set of nonnegative real numbers or scalars $t \in \mathbb{R}_+$.*

Therefore, evolutions are functions $x(\cdot) : t \in \mathbb{T} \mapsto x(t) \in X$ *describing the evolution of the state $x(t)$. The* trajectory (or orbit) *of an evolution $x(\cdot)$ is the subset $\{x(t)\}_{t \in \mathbb{T}} \subset X$ of states $x(t)$ when t ranges over $\mathbb{T}$.*

Unfortunately, for discrete time evolutions, tradition imposes upon us to regard discrete evolutions as sequences and to use the notation $\overrightarrow{x} : j \in \mathbb{N} \mapsto x_j := x(j) \in X$. We shall use this notation when we deal explicitly with discrete time. We use the notation $x(\cdot) : t \in \mathbb{R}_+ \mapsto x(t) \in X$ for continuous time evolutions and whenever the results we mention are valid for both continuous and discrete times. The context will tell the reader whether $x(\cdot)$ denotes an evolution when time ranges over $\mathbb{T}$ (either discrete or continuous) or when time ranges over $\mathbb{R}_+$.

We shall investigate both discrete and continuous time systems. Actually, the results dealing with viability kernels and capture basins use the same proofs. Only the tangential characterization becomes dramatically simpler, not to say trivial, in the case of discrete systems.

However, for computational purposes, we shall approximate continuous time systems by discrete time ones where the time scale becomes infinitesimal.

Warning: *Viability properties of the discrete analogues of continuous-time systems can be drastically different*: we shall see on the simple example of the Verhulst logistic equation that the interval $[0, 1]$ is invariant under the continuous system

$$x'(t) = rx(t)(1 - x(t))$$

whereas the viability kernel of $[0, 1]$ under its discrete analog

$$x_{n+1} = rx_n(1 - x_n)$$

is a Cantor subset of $[0, 1]$ when $r > 4$. Discrete analogs of continuous time dynamical systems can be different from their discretizations, which, under the assumptions of convergence theorems, share the same properties than the continuous time systems.

Warning: The terminology "trajectory" is often used as a synonym of evolution, but inadequately: a trajectory is the range of an evolution.

We shall assume most of the time that

1. the state space is a finite dimensional vector space $X := \mathbb{R}^n$,
2. evolutions are *continuous* functions $x(\cdot) : t \in \mathbb{R}_+ \mapsto x(t) \in X$ describing the evolution of the state $x(t)$.

We denote the space of continuous evolutions $x(\cdot)$ by $\mathcal{C}(0, \infty; X)$.

1.1.1 Stationary and Periodic Evolutions

We focus our attention on certain properties of evolutions, denoting by $\mathcal{H} \subset \mathcal{C}(0, \infty; X)$ the subset of evolutions satisfying these properties. For instance, the most common are stationary ones and periodic ones:

Definition 1.1.2. [***Stationary and Periodic Evolutions***]

1. *The subset* $\mathcal{X} \subset \mathcal{C}(0,\infty;X)$ *of* stationary evolutions *is the subset of evolutions* $x : t \mapsto x$ *when* x *ranges over the state space* X.

2. *The subset* $\mathcal{P}_T(X)$ *of* T-periodic evolutions *is the subset of evolutions* $x(\cdot) \in \mathcal{C}(0,\infty;X)$ *such that* $\forall t \geq 0,\ x(t+T) = x(t)$.

Stationary evolutions are periodic evolutions for all periods T.

Stationary and periodic evolutions have been the main topic of investigation in dynamical systems motivated by physical sciences. Indeed, the brain, maybe because it uses periodic evolutions of neurotransmitters through subsets of synapses, has evolved to recognize periodic evolutions, in particular those surrounding us in daily life. Their extensive study is perfectly legitimate in physical sciences, as well as their new developments (bifurcations, catastrophes, dealing with the dependence of equilibria in terms of a parameter, and chaos, investigating the absence of continuous dependence of evolution(s) with respect to the initial states, for instance).

However, even though we shall study evolutions regulated by constant parameters (passive evolutions), bifurcations are quite difficult to observe, as it was observed in section 3.3 of the book *Introduction to nonlinear systems and chaos* by Stephen Wiggins untitled *"On the Interpretation and Application of Bifurcation Diagrams: A Word of Caution"*: *At this point, we have seen enough examples so that it should be clear that the term bifurcation refers to the phenomenon of a system exhibiting qualitatively new dynamical behavior as parameters are varied. However, the phrase "as parameters are varied" deserves careful consideration... ln all of our analyses thus far the parameters have been constant. The point is that we cannot think of the parameter as varying in time, even though this is what happens in practice. Dynamical systems having parameters that change in time (no matter how slowly!) and that pass through bifurcation values often exhibit behavior that is very different from the analogous situation where the parameters are constant.*

Insofar as physical sciences privilege the study of stability or chaotic behavior around attractors, *transient* states have been neglected, although they pervade economic, social and biological evolutions.

1.1.2 Viable and Capturing Evolutions

Investigating evolutionary problems involving living beings should start with identifying the constraints bearing on the variables which cannot — or should not — be violated. Therefore, we consider mainly evolutions $x(\cdot)$ *viable in a subset* $K \subset X$ representing a environment (an environment) in which the trajectory of the evolution must remain forever:

$$\forall t \geq 0, \ \ x(t) \in K \tag{1.1}$$

or *capturing the target* C in the sense that they are viable in K until they reach the target C *in finite time*:

$$\exists\, T \geq 0 \text{ such that} \left\{ \begin{array}{l} x(T) \in C \\ \forall t \in [0,T], \ x(t) \ \in \ K \end{array} \right. \tag{1.2}$$

Definition 1.1.3. [***Viable and Capturing Evolutions***] *The subset of evolutions* viable *in* K *is denoted by*

$$\mathcal{V}(K) := \{x(\cdot) \mid \forall t \geq 0, \ \ x(t) \in K\} \tag{1.3}$$

and the subset of evolutions capturing *the target* C *by*

$$\mathcal{K}(K,C) := \{x(\cdot) \mid \exists\, T \geq 0 \textit{ such that}\, x(T) \in C \;\&\; \forall t \in [0,T], \ x(t) \ \in \ K\} \tag{1.4}$$

We also denote by

$$\mathcal{V}(K,C) := \mathcal{V}(K) \cup \mathcal{K}(K,C) \tag{1.5}$$

the set of evolutions viable in K outside C*, i.e. that are viable in* K *forever or until they reach the target* C *in finite time.*

Example: The first examples of environments used in control theory were vector (affine) subsets. Nonlinear control theory used first geometrical methods, which required *smooth equality constraints*, yielding constrained subsets of the form

$$K := \{x \mid g(x) = 0\}$$

These subsets, as well as more general manifolds (Klein bottle, for instance), having empty interiors, the viability and invariance problems were evacuated. This is no longer the case when the constrained subset is defined by inequality constraints, even smooth ones, yielding subsets of the form

$$K := \{x \mid g(x) \leq 0\}$$

the boundary of which is a proper subset. Subsets of the form

$$K := \{x \in L \mid g(x) \in M\}$$

where $L \subset X$, $g : X \mapsto Y$ and where $M \subset Y$ are typical constrained subsets encountered in mathematical economics. This is for such cases that mathematical difficulties appeared. ■

Constrained subsets in economics and biology are generally not smooth. The question arose to build a theory and forge new tools that did require neither the smoothness nor the convexity of the constrained subsets. *Set-valued analysis*, motivated in part by these viability and capturability issues, provided such tools.

Remark: These constraints can depend on time (time-dependent constraints), upon the state, the history (or the path) of the evolution of the state. Morphological equations are kind of differential equations governing the evolution of the constrained state $K(t)$ and can be paired with evolutions of the state. These topics are addressed later in these lecture notes.

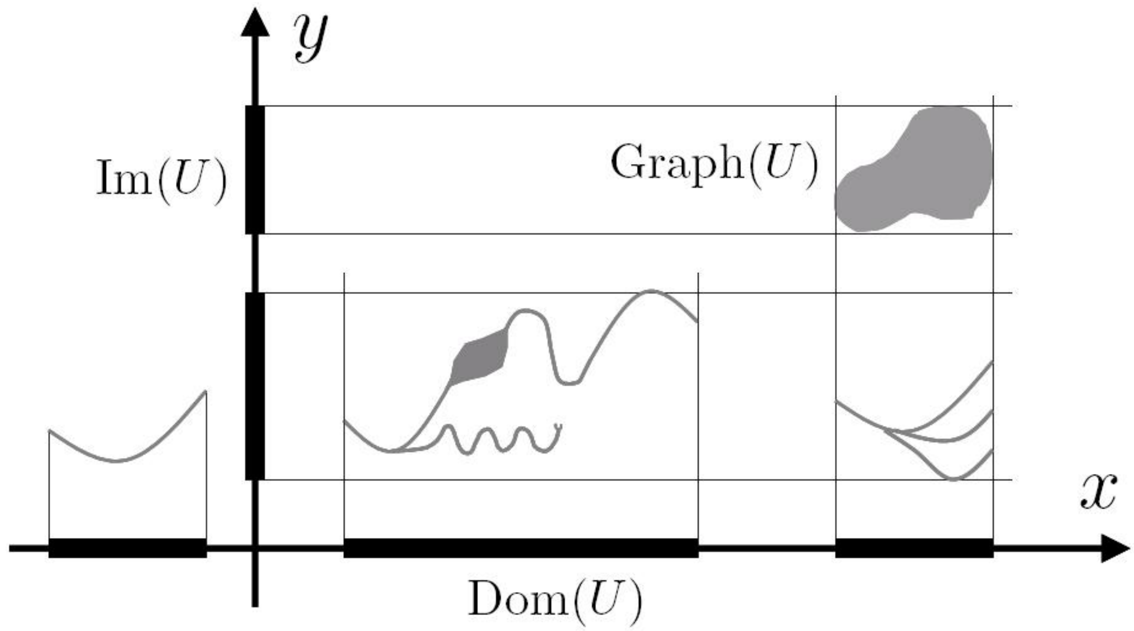
These constraints have to be confronted with evolutions. This is time to describe how these evolutions are produced and to design mathematical translations of several evolution art mechanisms.

1.2 Set-Valued Maps

We begin by defining set-valued maps:

Definition 1.2.1. [***Set-Valued Map***] *A set-valued map $F : X \rightsquigarrow Y$ associates with any $x \in X$ a subset $F(x) \subset Y$ (which may be the empty set $\emptyset$). It is a (single-valued) map if for any x, $F(x)$ is reduced to an element y. The graph* Graph(F) *os a set-valued map F is the set of pairs $(x, y) \in X \times Y$ satisfying $y \in F(x)$. If F is a single-valued map, it coincides with the usual concept of graph. Its* domain $\mathrm{Dom}(F)$ *is the subset of elements $x \in X$ such that $F(x)$ is not empty and its* image $\mathrm{Im}(F) = \bigcup_{x \in X} F(x)$ *is the union of the values $F(x)$ of F when x ranges over X. The* inverse F^{-1} *of F is the set-valued map from Y to X defined by*

$$x \in F^{-1}(y) \iff y \in F(x) \iff (x, y) \in \mathrm{Graph}(F)$$



1. Graph of a set-valued map

Definition 1.2.2. [***Selections***] *A* selection *of a set-valued map $U : X \rightsquigarrow \mathcal{U}$ is a single-valued map $\widetilde{u} : x \mapsto \widetilde{u}(x)$ such that*

$$\forall x, \;\; \widetilde{u}(x) \in U(x)$$

1.3 Discrete Systems

Discrete evolutionary systems can be defined on any metric state space X.

Examples of State Spaces for Discrete Systems:

1. When $X := \mathbb{R}^d$, we take any of the equivalent vector space metrics for which the addition and the multiplication by scalars is continuous.
2. When $X_\rho := \rho\mathbb{Z}^d$ is a grid with step size ρ, we take the discrete topology, defined by $d(x, x) := 0$ and

$d(x, y) := 1$ whenever $x \neq y$. A sequence of elements $x_n \in X$ converges to x if there exists an integer N such that for any $n \geq N$, $x_n = x$, any subset is both closed and open, the compacts are finite subsets. Any single-valued map from some space E to $X := \mathbb{Z}^d$ is continuous.

Deterministic discrete systems

$$\forall j \geq 0,\ x_{j+1} = \varphi(x_j) \text{ where } \varphi : x \in X \mapsto \varphi(x) \in X$$

are the simplest ones to formulate, but not necessarily the easiest ones to investigate. They have attracted the attention of many mathematicians.

Definition 1.3.1. [***Evolutionary Systems associated with Discrete Systems***] *Let X be any metric space and $\varphi : X \mapsto X$ be the single-valued map associating with any state $x \in X$ its successor $\varphi(x) \in X$.*
The space of discrete evolutions $\overrightarrow{x} := \{x_j\}_{j \in \mathbb{N}}$ is denoted by $X^{\mathbb{N}}$. The evolutionary system *$\mathcal{S}_\varphi : X \mapsto X^{\mathbb{N}}$ defined by the $\varphi : X \mapsto X$ is the map associating with any $x \in X$ the set $\mathcal{S}_\varphi(x)$ of discrete evolutions $\overrightarrow{x}$ starting at $x_0 = x$ and governed by the discrete system*

$$\forall j \geq 0,\ x_{j+1} = \varphi(x_j)$$

An equilibrium of a discrete dynamical system is a stationary evolution governed by this system.

An equilibrium $\overline{x} \in X$ (stationary point) of an evolution $\overrightarrow{x}$ by governed by the discrete system $x_{j+1} = \varphi(x_j)$ is a *fixed point* of the map φ, i.e., a solution he equation $\varphi(\overline{x}) = \overline{x}$. There are two families of Fixed Point Theorems based

1. either on the simple Banach-Cacciopoli-Picard Contraction Mapping Theorem in complete metric spaces,

2. or on the very deep and difficult 1910 Brouwer Fixed Point Theorem on convex compact subsets, the cornerstone of nonlinear analysis.

1.4 Differential Equations

1.4.1 Determinism and Predictability

Although these lecture notes is essentially dedicated to nondeterministic systems, we begin by the simplest of evolutionary systems which are associated with systems of differential equations

$$x'(t) = f(x(t))$$

where $f : X \mapsto X$ is the single-valued map associating with any state $x \in X$ its velocity $f(x) \in X$.

Definition 1.4.1. [***Evolutionary Systems associated with Differential Equations***] *Let $f : X \mapsto X$ be the single-valued map associating with any state $x \in X$ its velocity $f(x) \in X$.*
The evolutionary system *$\mathcal{S}_f : X \rightsquigarrow \mathcal{C}(0, \infty; X)$ defined by $f : X \mapsto X$ is the set-valued map associating with any $x \in X$ the set $\mathcal{S}_f(x)$ of evolutions $x(\cdot)$ governed by systems of differential equations*

$$x'(t) = f(x(t))$$

The evolutionary system is said to be deterministic *if $\mathcal{S}_f : X \rightsquigarrow \mathcal{C}(0, \infty; X)$ is single-valued.*
An equilibrium of a differential is a stationary solution of this equation.

The evolutionary system $\mathcal{S}_f$ associated with the single-valued map f is a priori a set-valued map, taking

1. nonempty values $\mathcal{S}_f(x)$ whenever there exists a solution to the differential equation starting at x, guaranteed by (local) existence theorems (the Peano Theorem, when f is continuous),
2. at most one value $\mathcal{S}_f(x)$ whenever uniqueness of the solution starting at x is guaranteed.

There are many sufficient conditions guaranteeing the uniqueness : f is Lipschitz, by the Cauchy-Lipschitz Theorem, or f is monotone in the sense that there exists a constant $\lambda \in \mathbb{R}$ such that

$$\forall x, y \in X, \quad \langle f(x) - f(y), x - y \rangle \le \lambda \|x - y\|^2$$

(we shall not review other uniqueness conditions here.)

Although a differential equation assigns a unique velocity to each state, this does not imply that the associated evolutionary system $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0,\infty;X)$ is *deterministic*, in the sense that it is univoque (single-valued). It may happen that several evolutions governed by a differential equation start from a same initial state.

The lack of uniqueness of some differential equations does not allows us to regard differential equations as a model of deterministic evolution. Determinism can be translated by evolutionary systems which associate with any initial state one and only one evolution.

An equilibrium $\overline{x}$ (stationary point) of an evolution governed by differential equation $x'(t) = f(x(t))$ being constant, its velocity is equal to 0, so that it is characterized as a solution to the equation $f(\overline{x}) = 0$.

Since the study of such equations, linear and nonlinear, has been for a long time been a favorite topic among mathematicians, the study of dynamical systems as for a long time focussed on equilibria : existence, uniqueness, stability.

Some nonlinear *differential equations* produce *chaotic* behavior, quite unstable and sensitive to initial conditions. However, for many problems arising in biological, cognitive, social and economic sciences, we face a completely *orthogonal situation*, governed by *differential inclusions, regulated or control systems, tychastic or stochastic systems*, but producing evolutions as *regular or stable* (in a very loose sense) as possible for the sake of adaptation and viability required for life.

1.4.2 Example: The Lorenz System

Since uncertainty is the underlying theme of these lecture notes, we propose to investigate the Lorenz system of differential equations, which are deterministic, but practicably unpredictable, as a simple example to test results presented in these lecture notes.

Lorenz introduced the following variables

1. x, proportional to the intensity of convective motion,
2. y, proportional to the temperature difference between ascending and descending currents,
3. z, proportional to the distortion (from linearity) of the vertical temperature profile.

Their evolution is governed by the following system of differential equations:

$$\left\{ \begin{array}{ll} (i) & x'(t) = \sigma y(t) - \sigma x(t) \\ (ii) & y'(t) = rx(t) - y(t) - x(t)z(t) \\ (iii) & z'(t) = x(t)y(t) - bz(t) \end{array} \right. \quad (1.6)$$

where $\sigma > b + 1$.

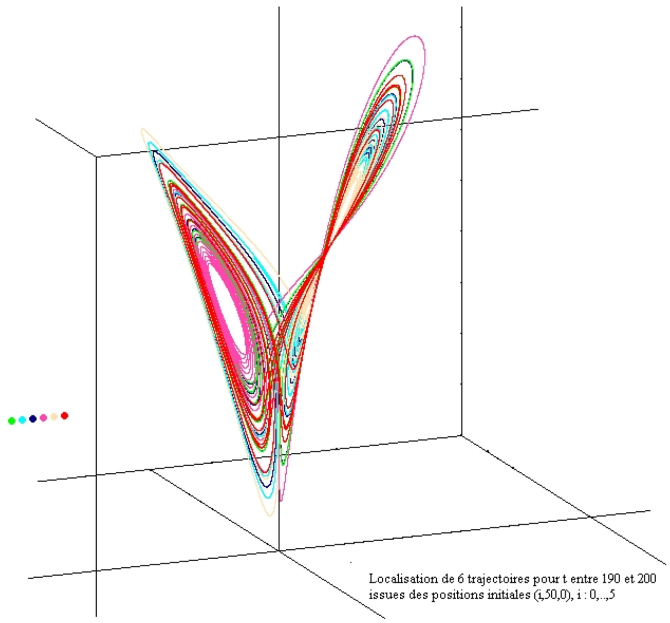


Figure 1.1: [Trajectories of six evolutions]
starting from initial conditions $(i, 50, 0)$, $i = 0, \ldots, 5$.*.Only the part of the trajectories from step times ranging between* 190 *and* 200 *are shown for clarity.*

We observe that the vertical axis $(0, 0, z)_{z\in\mathbb{R}}$ is a symmetry axis, which is also the viability kernel of the hyperplane $(0, y, z)$ under the Lorenz system, from which the solutions boil down to the exponentials $(0, 0, ze^{-bt})$.

Parameter r is the normalized Rayleigh number. If $r \in]0, 1[$, then 0 is an asymptotically stable equilibrium. If $r = 1$, the equilibrium 0 is "neutrally stable". When $r > 1$, the equilibrium 0 becomes unstable and two more equilibria appear:

$$e_1 := \left(\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1\right) \ \& \ e_2 := \left(-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1\right)$$

They are stable when

$$1 \ < \ r^\star := \frac{\sigma(\sigma + b + 3)}{\sigma - b - 1}$$

and unstable when $r > r^\star$. We take $\sigma = 10$, $b = \frac{8}{3}$ and $r = 28$ in the numerical experiments.

The dimension of the local stable manifold of the origin is equal to 2 and the local unstable manifold of the origin is one-dimensional. The flow contracts volumes at an exponential rate, since the divergence of the dynamics is equal to

$$\frac{\partial}{\partial x}\sigma(y-x)+\frac{\partial}{\partial y}(rx-y-xz)+\frac{\partial}{\partial z}(xy-bz)=-(\sigma+b+1)$$

so that the volume $\mathrm{Vol}(C(t))$ of the flow of any subset C with nonempty interior satisfies

$$\mathrm{Vol}(C(t))=\mathrm{Vol}(C)e^{-(\sigma+b+1)t}\leq \mathrm{Vol}(C)e^{-13t} \text{ for the chosen values}$$

thanks to the Stokes formula. The volumes are contracted very fast to the volume of the attractor which is equal to 0.

1.5 Regulons and Tyches

For systems involving living systems, agents interfering with the evolutionary mechanisms are often *myopic, conservative, lazy and opportunistic*, from molecules to (wo)men, enjoying some *contingent* freedom to choose among some *regulons* (regulatory parameters) to govern evolutions, in order to protect themselves against *tychastic* uncertainty, obeying no statistical regularity. This translated by adding to state variables other ones, controls, regulons and tyches, the names describing the questions concerning their role in the dynamics of the system.

State variables, constituting the components of the state of the system, evolve according to evolutionary laws involving several variables, called *parameters*, which may in their turn depend on *observation variables* of the states:

Classification of Variables:

1. *state variables*,
2. *parameters*,
3. *Observation variables*, such as measures, information, predictions, etc., given or built.

In *control theory* (i.e., *automatics*, and, when applied to mechanical systems, *robotics*), parameters have to be chosen in order to solve some specific requirements (optimality, reachability) by an actor (agent, decision-maker, etc.). However, even control theory has to take into account some uncertainty (disturbances, perturbations, etc.). Observations are usually *measurements* of state variables gathered along time.

In social and economic sciences, as well as in control theory, these parameters are not under the control of an agent involved in the evolutionary mechanism governing the evolutions of the state of the system under investigation, taking into account observations resulting themselves from the evolution of the state.

We distinguish several categories of parameters, according to the existence or the absence of an actor (controller, agent, decision-maker, etc.) acting on them on one hand, or the degree of knowledge or control on the other hand, and to explain their role:

Classification of Parameters Parameters controlled by an actor:

1. *control parameters* (or decision parameters)
2. *others*:
   (a) *regulons or regulation controls*
   (b) *tyches, perturbations, disturbances, random events*

They participate in different ways to the general concept of *uncertainty*.

In physics and engineering, the actors are well identified and their purpose clearly defined, so that only state, control and observation variables matter.

In the so called "soft sciences" involving uncertain evolutions of systems (organizations, organisms, organs, etc.) of living beings, the situation is more complex, because the identification of actors governing the evolution of parameters is more questionable, so that we regard in this case these parameters as regulons (regulatory parameters).

Examples of States and Regulons

Field	**State**	**Regulon**	**Viability**	**Actors**
Economics	physical goods	fiduciary goods	economic scarcity	agents
Genetics	phenotype	genotype	viability or homeostasis	bio-mechanical metabolism
sociology	psychological state	cultural codes	sociability	individuals
cognitive sciences	sensorimotor states	conceptual codes	adaptiveness	organims

1.6 Discrete Nondeterministic Systems

Here, the time set is $\mathbb{N}$, the state space is any metric set X and the evolutionary space is the space $X^{\mathbb{N}}$ of sequences $\overrightarrow{x} := \{x_j\}_{j\in\mathbb{N}}$ of elements $x_j \in X$. The space of parameters (controls, regulons or tyches) is another set $\mathcal{U}$. The evolutionary system is defined by the discrete parameterized system (φ, U) where

1. $\varphi : X \times \mathcal{U} \mapsto X$ is a map associating with any state-parameter pair (x, u) the next state $\varphi(x, u)$,
2. $U : X \rightsquigarrow \mathcal{U}$ is a set-valued map associating with any state x a set $U(x)$ of parameters *feeding back* on the state x.

Definition 1.6.1. [***Discrete Systems with State-Dependent Parameters***] *A discrete parameterized system (φ, U) defines the evolutionary system $\mathcal{S}_\Phi : X \rightsquigarrow X^{\mathbb{N}}$ in the following way: for any $x \in X$, $\mathcal{S}_\Phi(x)$ is the set of sequences $\overrightarrow{x}$ governed by*

$$\left\{ \begin{array}{ll} (i) & x_{j+1} = \varphi(x_j, u_j) \\ (ii) & u_j \in U(x_j) \end{array} \right. \tag{1.7}$$

starting from x.

When the parameter space is reduced to $\{0\}$, we find difference equations $x_{j+1} = \varphi(x_j)$ as a particular case. They generate deterministic evolutionary systems $\mathcal{S}_\varphi : X \mapsto X^{\mathbb{N}}$.

Setting

$$\Phi(x) := \varphi(x, U(x)) = \{\varphi(x, u)\}_{u \in U(x)}$$

the subset of all available successors $\varphi(x, u)$ at x when u ranges over the set of parameters allows us to treat these dynamical systems as difference inclusions:

Definition 1.6.2. [***Difference Inclusions***] *Let $\Phi(x) := \varphi(x, U(x))$ denote the set of velocities of the parameterized system. The evolutions $\overrightarrow{x}$ governed by the parameterized system*

$$\left\{ \begin{array}{ll} (i) & x_{j+1} = \varphi(x_j, u_j) \\ (ii) & u_j \in U(x_j) \end{array} \right. \tag{1.8}$$

are governed by the difference inclusion

$$x_{j+1} \in \Phi(x_j) \tag{1.9}$$

and conversely.

An equilibrium of a difference inclusion is a stationary solution of this inclusion.

Actually, any difference inclusion $x_{j+1} \in \Phi(x_j)$ can be regarded as a parameterized system (φ, U) by taking $\phi(x,u) := u$ and $U(x) := \Phi(x)$.

Selections of the set-valued map U are *retroactions* (see Definition 1.3, p.28) governing specific evolutions.

Among them, we single out the *heavy retroaction*:

Consider the set-valued map $s : \mathcal{P}(\mathcal{U}) \times \mathcal{U} \rightsquigarrow \mathcal{U}$ associating with any pair (A, u) the subset $s(A,u) := \{v \in A \mid d(u,v) = \inf_{w \in A} d(u,w)\}$ of "best approximations of u by elements of A". For instance, when the state space is a finite dimensional vector space X supplied with a scalar product and when the subsets $U(x)$ are closed and convex, the projection theorem implies that the map $s(U(x), u)$ is single-valued.

The evolutions governed by the dynamical system

$$\forall n \geq 0, \; x_{n+1} \in \varphi(x_n, s(U(x_n), u_{n-1}))$$

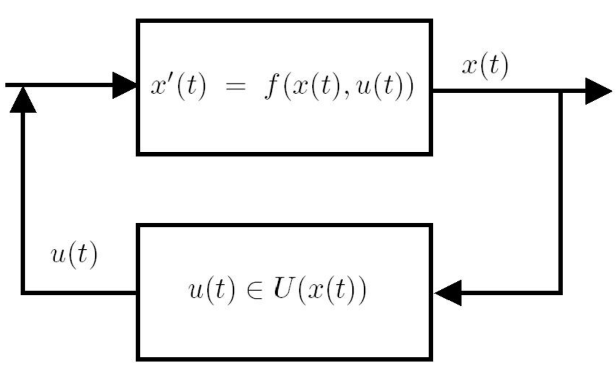
are called *heavy evolutions*.

This amounts to taking at time n a regulon $u_n \in s(U(x_n), u_{n-1})$ as close as possible as the regulon u_{n-1} chosen at the preceding step. If such a regulon u_{n-1} belongs to $U(x_n)$, it can be kept at the present step n. This in this sense that the selection $s(U(x), u)$ provides heavy solution, since the regulons are kept constant during the evolution as long as the viability is not at stakes.

1.7 Parameterized Dynamical Systems and Retroactions

1.7.1 Parameterized Dynamical Systems

The space of parameters (controls, regulons or tyches) is another finite dimensional vector space $\mathcal{U} := \mathbb{R}^c$.



Let $\mathcal{U} := \mathbb{R}^c$ be a space of parameters. A parameterized system is made of two "boxes":
1 - The "input-output box" *associating with any evolution $u(\cdot)$ of the parameter (*input*) the evolution governed by differential equation $x'(t) = f(x(t), u(t))$ starting from an initial state* (open loop),
2 - The non deterministic "output-input box"*, associating with any state a subset $U(x)$ of parameters (*output*).*

Figure 1.2: [Parameterized Systems]

The *evolutionary system* $\mathcal{S} : X \leadsto \mathcal{C}(0,\infty;X)$ defined by the control system (f,U) is the set-valued map associating with any $x \in X$ the set $\mathcal{S}(x)$ of evolutions $x(\cdot)$ governed by the control (or regulated) system

$$\begin{cases} (i) & x'(t) = f(x(t),u(t)) \\ (ii) & u(t) \in U(x(t)) \end{cases} \tag{1.10}$$

starting from x.

The parameters range over a state-dependent *cybernetic map* $U : x \leadsto U(x)$, providing the system opportunities to adapt at each state to viability constraints (often, as slowly as possible) and/or to regulate intertemporal optimal evolutions.

Remark: Differential equation (1.10)i) is an "input-output map" associating an output-state with an input-control. Inclusion (1.10)ii) associates input-controls with output-states, *"feeds back"* the system (the a priori feedback relation is set-valued, otherwise, we just obtain a differential equation). See Figure 1.2, p.26. ■

The study of parameterized systems (1.10) depends on the interpretation of the parameters, either regarded as controls and regulons on one hand, or as tyches or random variables on the other one.

When parameters are either controls or regulons, we are interested in regulating the system in the sense that we are looking for **at least one evolution** of the evolutionary system satisfying an evolutionary property, one can regard the evolutionary system as a *control system* or a *cybernetic system* (from the Greek *kubernesis*,"control","govern", as it was suggested first by André Ampère (1775-1836), and then, by *Norbert Wiener* (1894-1964).

When parameters represent tyches (disturbances, perturbations, etc.), we are interested in "robust" control of the system in the sense that **all evolutions** of the evolutionary system starting from a given initial state satisfy a given evolutionary property. Such an evolutionary system can be regarded as a *tychastic system*. These evolutions that are not under the control of a controller or a decision-maker could be called "random evolutions" if this vocabulary was not already confiscated by probabilists. To describe this situation, we suggest to borrow the concept of *tyche* from *Charles Peirce* (1839-1914), and to call in this case the control system a *tychastic system*

In summary, the questions that emerge when investigating the evolution of systems depend upon the role played by the parameters:

1. control or regulon, involving *cybernetic properties*,

2. tyches or random events, involving *tychastic or stochastic properties*,

3. both regulons and tyches, as in *dynamical games* or *tychastic control systems*.

1.7.2 Retroactions

In control theory, open and closed loop controls, feedbacks or retroactions provide the central concepts of cybernetics and general systems theory:

Definition 1.7.1. [***Retroactions***] *Retroactions are single-valued maps* $\widetilde{u} : (t, x) \in \mathbb{R}_+ \times X \mapsto \widetilde{u}(t, x) \in \mathcal{U}$ *that are plugged in the differential equation*

$$x'(t) = f(x(t), \widetilde{u}(t, x(t))) \tag{1.11}$$

In control theory, state-independent retroactions $t \mapsto \widetilde{u}(t, x) := u(t)$ *are called* open loop controls *whereas time-independent retroactions* $x \mapsto \widetilde{u}(t, x) := \widetilde{u}(x)$ *are called* closed loop controls*. See Figure 1.3, p.28.*

The theorems characterizing the viability kernel of an environment of the capture basin of a target provide also a regulation map $x \rightsquigarrow R(x) \subset U(x)$ regulating evolutions viable in K:

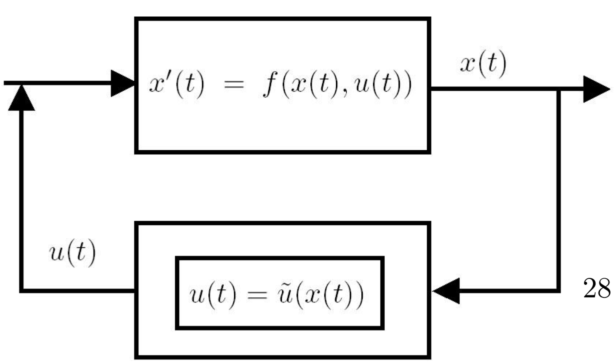
Definition 1.7.2. [**Regulation Map**] *Let us consider control system*

$$\begin{cases} (i) & x'(t) = f(x(t), u(t)) \\ (ii) & u(t) \in U(x(t)) \end{cases}$$

and an environment K. A set-valued map $x \rightsquigarrow R(x) \subset U(x)$ *is called a* regulation map governing viable evolutions *if the viability kernel of K is invariant under the control system*

$$\begin{cases} (i) & x'(t) = f(x(t), u(t)) \\ (ii) & u(t) \in R(x(t)) \end{cases}$$

Actually, we are looking for single-valued regulation maps governing viable evolutions, which are usually called feedbacks:



The single-valued maps $x \mapsto \widetilde{u}(x)$ *are called the* feedbacks *(or servomechanisms, closed loop controls, etc.) allowing to pilot evolutions by using controls of the form* $u(t) := \widetilde{u}(x(t))$ *in system (1.10), p.27: Knowing such a feedback, the evolution is governed by ordinary differential equation* $x'(t) = f(x(t), \widetilde{u}(x(t)))$

Figure 1.3: [**Feedbacks**]

Hence, knowing the regulation map R, *viable feedbacks* are single-valued regulation maps.

The class $\widetilde{\mathcal{U}}$ in which retroactions are taken must be consistent with the properties of the parameterized system so that

- the differential equations $x'(t) = f(x(t), \widetilde{u}(t, x(t)))$ have solutions,
- for every $t \geq 0$, $\widetilde{u}(t, x) \in U(x)$

When no state-dependent constraints bear on the controls, i.e., when $U(x) = U$ does not depend on the state x, then open loop controls can be used to parameterize the evolutions $S(x, u(\cdot))(\cdot)$ governed by differential equations (1.10)i).

This is no longer the case when the constraints on the controls depend on the state. In this case, we parameterize the evolutions of the control system (1.10) by closed loop controls or retroactions.

Inclusion (1.10)ii), which associates input-controls with output-states, "feeds back" the system in a set-valued way. The question arises whether one can select single-valued retroactions in the set-valued map $U : X \rightsquigarrow \mathcal{U}$.

The choice of an adequate class $\widetilde{\mathcal{U}}$ of feedbacks regulating specific evolutions satisfying required properties is often an important issue. Finding them may be a difficult problem to solve. Even though one could solve this problem, computing or using a feedback in a class too large may not be desirable whenever feedbacks are required to belong to a class of specific maps (constant maps, time-dependent polynomials, etc.). Another issue concerns the use of a prescribed class of retroactions and to "combine" them to construct new feedbacks for answering somme questions, viability or capturability, for instance.

Remark: In one-dimensional systems, retroactions are classified in positive retroactions, when the phenomenon is "amplified", and negative ones in the opposite case. ■

The concept of retroaction plays a central role in control theory, for building servomechanisms, and then, later, in all versions of the "theory of systems" born from the influence of the mathematics of their time on biology. The fact that not only effects resulted from causes, but that also effects retroacted on causes, "closing" a system, has had a great influence in many fields.

1.7.3 Differential Inclusions

When the constraints bearing on the parameters (controls, regulons, tyches) are state dependent, we can no longer use differential equations.

Indeed, denoting by

$$F(x) := f(x, U(x)) = \{f(x,u)\}_{u \in U(x)}$$

the subset of all available velocities $f(x,u)$ at x when u ranges over the set of parameters, we observe that

Lemma 1.7.3. [***Differential Inclusions***] *Let $F(x) := f(x, U(x))$ denote the set of velocities of the parameterized system. The evolutions $x(\cdot)$ governed by the parameterized system*

$$\begin{cases} (i) & x'(t) = f(x(t), u(t)) \\ (ii) & u(t) \in U(x(t)) \end{cases} \tag{1.12}$$

are governed by the differential inclusion

$$x'(t) \in F(x(t)) \tag{1.13}$$

and conversely.

An equilibrium of a differential inclusion is a stationary solution of this inclusion.

By taking $f(x,u) := u$ and $U(x) := F(x)$, any differential inclusion $x'(t) \in F(x(t))$ appears as a parameterized system (f, U) parameterized by its velocities. Whenever we do not need to explicit the controls, it is simpler to consider a parameterized system as a differential inclusion.

Most theorems on differential equations can be adapted to differential inclusions (some of them, the basic ones, are indeed more difficult to prove), but they are by now available.

However, there are examples of differential inclusions without solutions, such as the simplest one:

Counter-Example: The constrained state is $K := [a,b]$ and the subsets of velocities are singletons except at one point $c \in]a,b[$, where $F(x) := \{-1, 1\}$:

$$F(x) \; := \begin{cases} +1 & \text{if} \quad x \in [a, c[\\ -1 \text{ or } +1 & \text{if} \quad x = c \\ -1 & \text{if} \quad x \in]c, b] \end{cases}$$

No evolution can start from c. Observe that this is no longer a counter-example when $F(c) := [-1, +1]$, since in this case c is an equilibrium, the velocity 0 belonging to $F(c)$. ■

Remark: Although a differential inclusion assigns several velocities to a same states, this does not imply that the associated evolutionary system is non deterministic. It may happen that for certain classes of differential inclusions. This is the case for instance when there exists a constant $\lambda \in \mathbb{R}$ such that

$$\forall x, y \in X,\ \forall u \in F(x),\ \forall v \in F(y),\ \ \langle u - v, x - y \rangle \leq \lambda \|x - y\|^2$$

because in this case evolutions starting from each initial state, if any, are unique. ■

For discrete dynamical systems, the single-valuedness of the dynamics $\varphi :\mapsto X$ is equivalent to the single-valuedness of the associated evolutionary system $\mathcal{S}_\varphi : X \mapsto X^{\mathbb{N}}$. This is no longer the case for continuous time dynamical systems:

Warning: The deterministic character of an evolutionary system generated by a parameterized system is concept different from the set-valued character of the map F. What matters is that the evolutionary system $\mathcal{S}$ associated with the parameterized system is single-valued (deterministic evolutionary systems) or set-valued (nondeterministic evolutionary systems).

1.8 Discretization Issues

The task for achieving this objective is divided in two different problems:

1. Approximate the continuous problem by discretized problem (in time) and digitalized on a grid (in state) by *difference inclusions* on *digitalized sets*. Most of the time, the real mathematical difficulties come from the proof of the convergence theorems stating that the limits of the solutions to the approximate discretized/digitalized problems converge (in an adequate sense) to solutions to the original continuous-time problem.

2. Compute the viability kernel or the capture basin of the discretized/digitalized problem with a specific algorithm, providing also the viable evolutions.

Let h denote the time discretization step (also called *cadence*). There are many more or less sophisticated ways to discretize a continuous parameterized system (f, U) by a discrete one (ϕ_h, U). The simplest way is to choose the explicit scheme $\phi_h(x, u) := x + hf(x, u)$. Indeed, the discretized system (1.7) can be written as

$$\frac{x_{j+1} - x_j}{h} = f(x_j, u_j) \text{ where } u_j \in U(x_j)$$

The simplest way to digitalize a vector space $X := \mathbb{R}^d$ is to embed a (regular) *grid*[1] $X_\rho := \rho\mathbb{Z}^d$ in X. Points of the grid are of the form $x := (\rho n_i)_{i=1,\ldots,n}$ where for all $i = 1, , \ldots, n$, n_i range over the set $\mathbb{Z}$ of positive or negative integers.

We cannot define the above discrete system on the grid X_ρ, because there are no reason why for any $x \in X_\rho$, $\phi_h(x,u)$ would belong to the grid X_ρ. Let us denote by $B := [-1,+1]^d$ the unit square ball of X^d. One way to overcome this difficulty is to "add" the set $\rho B = [-\rho,+\rho]^d$ to $\phi_h(x,u)$. Setting $\lambda A + \mu B := \{\lambda x + \mu y\}_{x\in A,\, y\in B}$ when $A \subset X$ and $B \subset X$ are nonempty subsets of a vector space X, we obtain the following example:

Definition 1.8.1. [***Explicit Discrete/Digital Approximation***] *zzz Parameterized control systems*

$$x'(t) \;=\; f(x(t),u(t)) \text{ where } u(t) \in U(x(t))$$

can be approximated by discrete/digital parameterized systems

$$\frac{x_{j+1}-x_j}{h} \in f(x_j,u_j) + \rho B \text{ where } u_j \in U(x_j)$$

which is a discrete system $x_{j+1} \in \Phi_\rho(x_j)$ on X_ρ where

$$\Phi_{h,\rho}(x) \;:=\; x + hf(x,U(x)) + \rho B$$

We can also use implicit difference schemes:

Definition 1.8.2. [***Implicit Discrete/Digital Approximation***] *Parameterized control systems*

$$x'(t) \;=\; f(x(t),u(t)) \text{ where } u(t) \in U(x(t))$$

can be approximated by discrete/digital parameterized systems

$$\frac{x_{j+1}-x_j}{h} \in f(x_{j+1},u_{j+1}) + \rho B \text{ where } u_{j+1} \in U(x_{j+1})$$

which is a discrete system $x_{j+1} \in \Psi_\rho(x_j)$ on X_ρ where

$$\Psi_{h,\rho}(x) \;:=\; (\mathbf{I} - hf(\cdot,U(\cdot)))^{-1}(x + \rho h B)$$

[1]supplied with the metric $d(x,y)$ equal to 0 if $x = y$ and to 1 if $x \neq y$.

Remark: How Tangential Conditions Emerge
Writing the viability conditions for the explicit finite-difference scheme to yield at least one evolution viable in K amounts to saying that for every $x \in K$, there exists $u \in U(x)$ such that $hf(x,u) \in K$, i.e.,

$$\forall x \in K, \ \exists \, u \in U(x) \mid f(x,u) \in \frac{K-x}{h}$$

The Bouligand-Severi tangent cone $T_K(x)$ (see Definition 5.7.1, p.170) to K at $x \in K$ is the upper limit in the sense of Painlevé-Kuratowski upper limit of $\frac{K-x}{h}$ when $h \to 0$: $f(x,u)$ is the limit of elements v_n such that $x + h_n v_n \in K$ for some $h_n \to 0+$, i.e., of velocities $v_n \in \dfrac{K-x}{h_n}$. ■

1.9 Evolutionary Systems

Therefore, we shall study general evolutionary systems defined as set-valued maps $X \rightsquigarrow \mathcal{C}(0,\infty;X)$ satisfying given requirements listed below. For continuous time evolutionary systems, the state space X is a finite dimensional vector space for most examples. However, besides the characterization of regulation maps, which are specific for control systems, many theorems are true even in cases when the evolutionary system is not generated by control systems or differential inclusions, and for infinite dimensional vector spaces X.

Examples of state spaces:

1. When $\mathbf{X} := \mathcal{C}(-\infty,0;X)$ is the space of evolution histories, we supply it with the metrizable compact convergence topology,
2. When X is a space of spatial functions when one deals with partial differential inclusions or distributed control systems, we endow it with its natural topology for which it is a complete metrizable spaces,
3. When $X := \mathcal{K}(\mathbb{R}^d)$ is the set of nonempty compact subsets of the vector space $\mathbb{R}^d$, we use the Pompeiu-Hausdorff topology (morphological and mutational equations. ■

The algebraic structures of the state space appear to be irrelevant, only the following algebraic structure of the evolutionary spaces $\mathcal{C}(0,\infty;X)$ are used in the properties of viability kernels and capture basins:

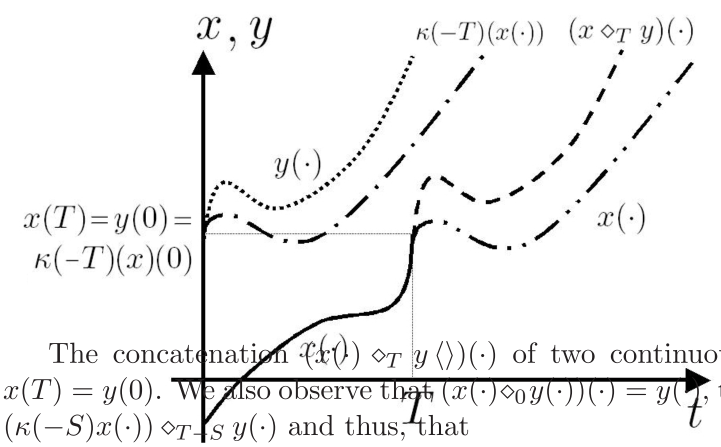
We shall perform the following operations on evolutions:

Definition 1.9.1. [***Translations and Concatenations***]

1. ***Translation*** *Let $x(\cdot) : \mathbb{R}_+ \mapsto X$ be an evolution. For all $T \geq 0$, the translation (to the left) $\kappa(-T)(x(\cdot))$ of the evolution $x(\cdot)$ is defined by $\kappa(-T)(x(\cdot))(t) := x(t+T)$,*

2. ***Concatenation*** *Let $x(\cdot) : \mathbb{R}_+ \mapsto X$ and $y(\cdot) : \mathbb{R}_+ \mapsto X$ be two evolutions. For all $T \geq 0$, the concatenation $(x(\cdot) \diamond_T y(\cdot))(\cdot)$ of the evolutions $x(\cdot)$ and $y(\cdot)$ at time T is defined by*

$$(x(\cdot) \diamond_T y(\cdot))(t) := \begin{cases} x(t) & \text{if} \quad t \in [0,T] \\ y(t-T) & \text{if} \quad t \geq T \end{cases} \tag{1.14}$$



- *plain (—): $x(\cdot)$ for $t \in [0,T]$;*
- *dash dot dot ($-\cdot\cdot$): $x(\cdot)$ for $t \geq T$;*
- *dot ($\cdots$): $y(\cdot)$;*
- *dash dot ($-\cdot$): $\kappa(-T)(x(\cdot))$;*
- *dashed ($-\ -$): $(x \diamond_T y)(\cdot)$.*

$x(\cdot)$ is thus the union of the plain and the dash dot dot. The concatenation $x(\cdot)\diamond_T y(\cdot)$ of x and y is the union of the plain and the dashed.

Figure 1.4: [**Translations and Concatenations**]

The concatenation $(x(\cdot) \diamond_T y\langle\rangle)(\cdot)$ of two continuous evolutions at time T is continuous if $x(T) = y(0)$. We also observe that $(x(\cdot)\diamond_0 y(\cdot))(\cdot) = y(\cdot)$, that $\forall T \geq S \geq 0$, $(\kappa(-S)(x(\cdot)\diamond_T y(\cdot))) = (\kappa(-S)x(\cdot)) \diamond_{T-S} y(\cdot)$ and thus, that

$$\forall T \geq 0, \ \kappa(-T)(x(\cdot) \diamond_T y(\cdot)) = y(\cdot)$$

The adaptation of these definitions to discrete time evolutions is obvious:

$$\begin{cases} (i) & \kappa(-N)(\overrightarrow{x})_j := x_{j+N} \\ (ii) & (\overrightarrow{x} \diamond_N \overrightarrow{y})_j := \begin{cases} x_j & \text{if} \quad 0 \leq j < N \\ y_{j-N} & \text{if} \quad j \geq N \end{cases} \end{cases} \tag{1.15}$$

We shall use only the following properties of evolutionary systems:

Definition 1.9.2. [***Evolutionary systems***] *Let us consider a set-valued map $\mathcal{S} : X \rightsquigarrow$*

$\mathcal{C}(0,\infty;X)$ *associating with each initial state* $x \in X$ *a (possibly empty) subset of* evolutions $x(\cdot) \in \mathcal{S}(x)$ *starting from* x *in the sense that* $x(0) = x$. *It is said to be an* evolutionary system *if it satisfies*

1. *the* translation property*: Let* $x(\cdot) \in \mathcal{S}(x)$*. Then for all* $T \geq 0$*, the translation* $\kappa(-T)(x(\cdot))$ *of the evolution* $x(\cdot)$ *belongs to* $\mathcal{S}(x(T))$*,*

2. *the* concatenation property*: Let* $x(\cdot) \in \mathcal{S}(x)$*. Then for every* $T \geq 0$ *and* $y(\cdot) \in \mathcal{S}(x(T))$*, the concatenation* $(x(\cdot) \diamond_T y(\cdot))(\cdot)$ *belongs to* $\mathcal{S}(x)$*.*

The evolutionary system is said to be deterministic *if* $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0,\infty;X)$ *is single-valued.*

There are several ways for describing continuity of the evolutionary system $x \rightsquigarrow \mathcal{S}(x)$ with respect to the initial state, regarded as stability property : Stability means generally that the solution of a problem depends continuously upon its data or parameters. Here, for differential inclusions, the data are usually and principally the initial states, but can also be other parameters involved in the right hand side of the differential inclusion. We shall introduce them later, when we shall study the topological properties of the viability kernels and capture basins.

1.10 Viability Kernels and Capture Basins for Discrete Time Systems

Definition 0.1.1, p.9 can be adapted to discrete evolutions $\overrightarrow{x}$: they are *viable in a subset* $K \subset X$ (an environment) if:

$$\forall n \geq 0, \ \ x_n \in K \tag{1.16}$$

and thy *capture* a target C if they are viable in K until it reaches the target C in *finite time*:

$$\exists \ N \geq 0 \text{ such that} \left\{ \begin{array}{l} x_N \in C \\ \forall n \leq N, \ x_N \ \in \ K \end{array} \right. \tag{1.17}$$

Consider a set-valued map $\Phi : X \rightsquigarrow X$ from a metric space X to itself, governing the evolution $\vec{x} : n \mapsto x_n$ defined by

$$\forall j \geq 0, \ x_{j+1} \in \Phi(x_j)$$

and the associated evolutionary system $\mathcal{S}_\Phi : X \rightsquigarrow X^{\mathbb{N}}$ associating with any $x \in X$ the set of evolutions $\vec{x}$ of solutions to the above discrete system starting at x. Replacing the space $\mathcal{C}(0,\infty;X)$ of continuous time-dependent functions by the space $X^{\mathbf{N}}$ of discrete-time dependent functions (sequences) and making the necessary adjustments in definitions, we can still regard $\mathcal{S}_\Phi$ as an evolutionary system from X to $X^{\mathbf{N}}$. The viability kernels $\text{Viab}_\Phi(K,C) := \text{Viab}_{\mathcal{S}_\Phi}(K,C)$ and the invariance kernels $\text{Inv}_\Phi(K,C) := \text{Inv}_{\mathcal{S}_\Phi}(K,C)$ are defined in the very same way:

Definition 1.10.1. [***Viability Kernel under a Discrete System***] *Let $K \subset X$ be a environment and $C \subset K$ a target.*

The subset $\text{Viab}_\Phi(K,C)$ *of initial states $x_0 \in K$ such that* ***at least one evolution*** $\overrightarrow{x} \in \mathcal{S}_\Phi(x_0)$ *starting at x_0 is viable in K for all $n \geq 1$ or viable in K until it reaches C in finite time is called the* viability kernel of K with target C *under $\mathcal{S}$.*

When the target $C = \emptyset$ is the empty set, we say that $\text{Viab}_\Phi(K) = \text{Viab}_\Phi(K,\emptyset)$ *is the* viability kernel *of K.*

The subset $\text{Capt}_\Phi(K,C)$ *of initial states $x_0 \in K$ such that* ***at least one evolution*** $\overrightarrow{x} \in \mathcal{S}_\Phi(x_0)$ *starting at x_0 is viable in K until it reaches C in finite time is called the* capture basin of C viable in K *under $\mathcal{S}_\Phi$.*

We say that

1. *a subset K is* viable outside the target $C \subset K$ *under the discrete system $\mathcal{S}$ if* $K = \text{Viab}_\Phi(K,C)$ *and that K is* viable *under $\mathcal{S}_\Phi$ if* $K = \text{Viab}_\Phi(K)$*,*
2. *that C is* isolated *in K if* $C = \text{Viab}_\Phi(K,C)$*,*
3. *that K is a* repeller *if* $\text{Viab}_\Phi(K) = \emptyset$*, i.e. if the empty set is isolated in K.*

We introduce the discrete invariance kernels and absorption basins:

Definition 1.10.2. [***Invariance Kernel under a Discrete System***] *Let $K \subset X$ be a environment and $C \subset K$ a target.*

The subset $\text{Inv}_\Phi(K,C) := \text{Inv}_{\mathcal{S}_\Phi}(K,C)$ *of initial states $x_0 \in K$ such that* ***all evolutions*** $\overrightarrow{x} \in \mathcal{S}_\Phi(x_0)$ *starting at x_0 are viable in K for all $n \geq 1$ or viable in K until they reach C in finite time is called the* discrete invariance kernel of K with target C *under $\mathcal{S}_\Phi$.*

When the target $C = \emptyset$ is the empty set, we say that $\text{Inv}_\Phi(K) := \text{Inv}_\Phi(K,\emptyset)$ *is the* discrete invariance kernel *of K.*

The subset $\text{Abs}_\Phi(K,C)$ *of initial states $x_0 \in K$ such that* ***all evolutions*** $\overrightarrow{x} \in \mathcal{S}_\Phi(x_0)$ *starting at x_0 are viable in K until they reach C in finite time is called the* absorption basin of K with target C *under $\mathcal{S}_\Phi$.*

We say that

1. *a subset K is* invariant outside a target $C \subset K$ *under the discrete system $\mathcal{S}_\Phi$ if $K := \mathrm{Inv}_\Phi(K, C)$ and that K is* invariant *under $\mathcal{S}_\Phi$ if $K = \mathrm{Inv}_\Phi(K)$,*

2. *that C is* separated *in K if $C = \mathrm{Inv}_\Phi(K, C)$.*

In the discrete-time case, the following characterization of viability and invariance of K outside a target $C \subset K$ is a tautology:

Theorem 1.10.3. [***The Discrete Viability and Invariance Characterization***] *Let $K \subset X$ and $C \subset K$ be two subsets and $\Phi : K \rightsquigarrow X$ govern the evolution of the discrete system. Then the two following statements are equivalent*

1. *K is viable outside C under Φ if and only if*

$$\forall x \in K \backslash C,\ \Phi(x) \cap K \neq \emptyset \tag{1.18}$$

2. *K is invariant outside C under Φ if and only if*

$$\forall x \in K \backslash C,\ \Phi(x) \subset K \tag{1.19}$$

Unfortunately, the analogous characterization is much more difficult in the case of continuous time control systems, where the proofs of the statements require almost all basic theorems of functional analysis to be proved.

Remark: The fact that the above characterizations of viability and invariance in terms of (1.18) and (1.19) are trivial does not imply that using them is necessarily an easy task: Proving that $\Phi(x) \cap K$ is not empty or that $\Phi(x) \subset K$ can be difficult and require some sophisticated theorems of nonlinear analysis. We shall meet the same obstacles — but compounded — when using the Viability and Invariance Theorems for continuous time systems. ■

Definition 1.10.4. [***Regulation Map***] *Let (φ, U) be a discrete parameterized system, K be an environment and $C \subset K$ be a target. The* regulation map R_K *is defined on the* *viability kernel of K by*

$$\forall x \in \mathrm{Viab}_{(\varphi,U)}(K,C) \setminus C, \quad R_K(x) := \{u \in U(x) \;\; \textit{such that}\; \varphi(x,u) \in \mathrm{Viab}_{(\varphi,U)}(K,C)\} \tag{1.20}$$

The regulation map is computed from the discrete parameterized system (φ, U), the environment K and the target $C \subset K$.

Proposition 1.10.5. [***Sub-Regulation Maps***] *The* regulation map $R_K \subset U$ *defined on the viability kernel* $\mathrm{Viab}_{(\varphi,U)}((K,C))$ *satisfies property*

$$\mathrm{Inv}_{(\varphi,R_K)}(K,C) \;=\; \mathrm{Viab}_{(\varphi,U)}(K,C) \tag{1.21}$$

For any submap $P_K \subset R_K$ with nonempty values, property

$$\mathrm{Inv}_{(\varphi,P_K)}(K,C) \;=\; \mathrm{Viab}_{(\varphi,U)}(K,C)$$

remains true

Viability Kernel Algorithms

For evolutionary systems associated with discrete dynamical inclusions and control systems, the *Viability Kernel Algorithm* and the *Capture Basin Algorithm* devised by Patrick Saint-Pierre allow us to

1. compute the viability kernel of an environment or the capture basin of a target under a control system,

2. compute the evolutions viable in the environment forever or until they reach the target in finite time.

Indeed, starting from an initial state in the viability kernel, standard algorithms for computing solutions (the so-called shooting methods) do not take into consideration the corrections *for imposing the viability of the solution*. Since the initial state is only in an approximation of the viability kernel, the absence of these corrections does not allow us to "tame" evolutions which quickly leave the environment, above all for systems which are sensitive to initial states, such as the Lorenz system.

This algorithm *manipulates subsets instead of functions*, and is part of the emerging field of *"set-valued numerical analysis"*. Since the viability kernel of a environment is the

the subset of initial states from which at least an evolution remains in the constrained set, "shooting methods", which make sense for differential equations, but not for differential inclusions, amount to checking whether initial states provide evolutions which remain in a given set for a long time (and not for ever). But there is no guarantee that the time chosen to stop the computation of the solution is large enough to approximate closely the viability kernels.

Since these algorithms "trap the tamed" few evolutions that are viable in these kernels or basins, even when the systems are chaotic or highly sensitive to initial states. This is quite important, because, even starting from an initial state in a viable subset, approximations provided with very precise schemes of solutions which should be viable in the set may actually leave it very quickly. The viability kernel algorithm provides the exact subset of initial states from which at least one evolution of the discretized/digitalized system remains forever in the constrained set. However, viability kernel and capture basin algorithms face the same "dimensionality curse" than algorithm for solving partial differential equations since they manipulate tables which become huge when the dimension of the state space is larger than 4 or 5.

Viability Kernels under the Quadratic Map

The quadratic map φ associates with $x \in [0,1]$ the element $\varphi(x) = rx(1-x) \in \mathbb{R}$, governing the celebrated *discrete logistic system* $= x_{j+1} = rx_j(1-x_j)$. The fixed points of φ are 0 and $c := \dfrac{r-1}{r}$, which is smaller than 1. We also observe that $\varphi(0) = \varphi(1) = 0$ so that the successor of 1 is the equilibrium 0.

For $X : [0,1] \subset \mathbb{R}$ to be a state space under this discrete logistic system, we need that φ maps $X : [0,1]$ to itself, i.e., that $r \leq 4$. Otherwise, for $r > 4$, the roots of the equation $\varphi(x) = 1$ are equal to $a := \dfrac{1}{2} - \dfrac{\sqrt{r^2 - 4rx}}{2r}$ and $b := \dfrac{1}{2} + \dfrac{\sqrt{r^2 - 4rx}}{2r}$, where $b < c$. We denote by $d \in [0,a]$ the other root of the equation $\varphi(d) = c$. Therefore, for any $x \in]a,b[$, $\varphi(x) > 1$.

A way to overcome this difficulty is to associate with the single-valued $\varphi : [0,1] \mapsto \mathbb{R}$ the set-valued map $\Phi : [0,1] \rightsquigarrow [0,1]$ defined by $\Phi(x) := \varphi(x)$ when $x \in [0,a]$ and $x \in [b,1]$ and $\varphi(x) := \emptyset$ when $x \in]a,b[$. The inverse Φ^{-1} is defined by

$$\Phi^{-1}(y) := \left(\omega^\flat(y) := \left(\frac{1}{2} - \frac{\sqrt{r^2 - 4ry}}{2r}\right), \omega^\sharp(y) := \frac{1}{2} + \frac{\sqrt{r^2 - 4ry}}{2r}\right)$$

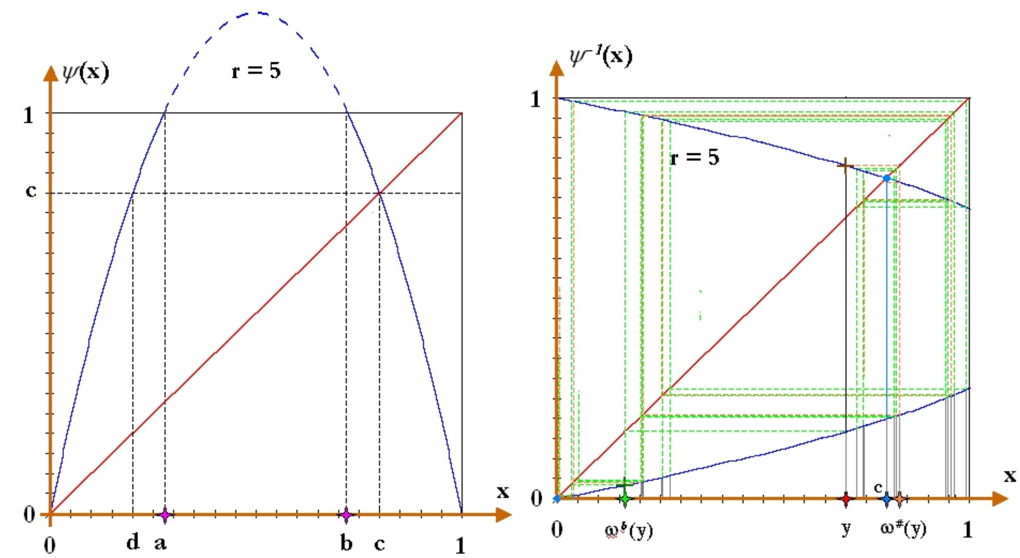


Figure 1.5: [**Discrete Logistic System**] *The graph of the function $x \mapsto \varphi(x) := rx(1-x)$ for $r = 5$ is displayed as a function $\varphi : [0,1] \mapsto \mathbb{R}$ as a set-valued map $\Phi : [0,1] \rightsquigarrow [0,1]$ associating with any $x \in [a,b]$ the empty set. Equilibria are the abscissas of points of the intersection of the graph* Graph(φ) *of φ and of the bisectrix. We observe that* 0 *and the point c (to the right of b) are equilibria. On the right, the graph of the inverse is displayed, with its two branches .*

The predecessors $\Phi^{-1}(0)$ and $\Phi^{-1}(c)$ of equilibria 0 and c are initial states of viable discrete evolutions because, starting from them, the equilibria are their successors, from which the evolution remains forever. They are made of $\omega^\sharp(0) = 1$ and of $c_1 := \omega^\flat(c)$. In the same way, the four predecessors $\Phi^{-2}(0) = \{\omega^\flat(1) = a, \omega^\sharp(1) = b\}$ and $\Phi^{-2}(c)$ are initial states of viable evolutions, since, after two iterations, we obtain the two equilibria from which the evolution remains forever. And so on: The subsets $\Phi^{-p}(0)$ and $\Phi^{-p}(c)$ are made of initial states from which star evolutions which reach the two equilibria after p iterations, and thus, which are viable in K. They belong to the *viability kernel* of K.

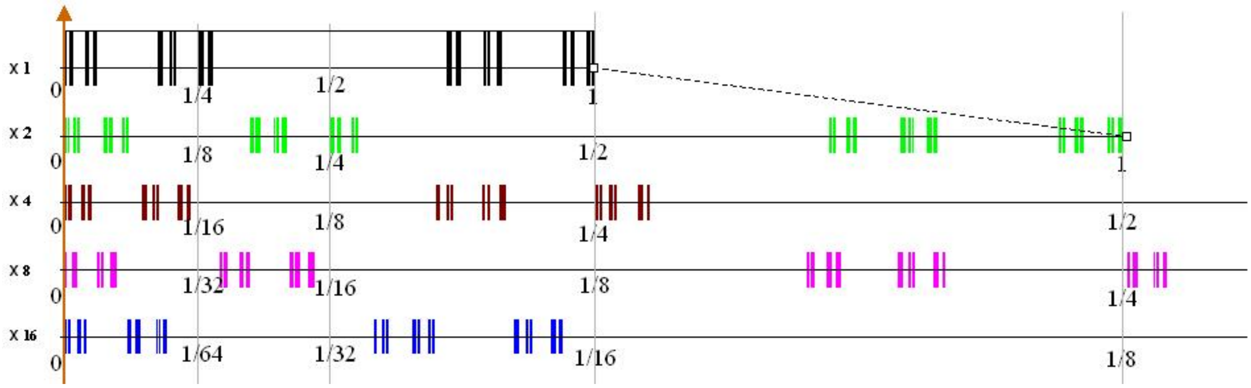


Figure 1.6: [**Viability Kernel under the Quadratic Map**]
The viability kernel of the interval $[0, 1]$ *under the quadratic map* Φ *associated with the map* $\varphi(x) := \{5x(1-x)\}$ *is a uncountable, symmetric Cantor set.*

1.11 Viability Kernels and Capture Basins for Continuous Time Systems

Let $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0, \infty; X)$ denote the evolutionary system associated with the parameterized dynamical system (1.10) and $\mathcal{H} \subset \mathcal{C}(0, \infty; X)$ be a subset of evolutions sharing a given property.

1.11.1 Definitions

When the parameterized system is regarded as a control system, we single out the inverse image of $\mathcal{H}$ under the evolutionary system:

Definition 1.11.1. [***Inverse Image under an Evolutionary System***] *Let* $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0, \infty; X)$ *denote an evolutionary system and* $\mathcal{H} \subset \mathcal{C}(0, \infty; X)$ *a subset of evolutions sharing a given property. The set*

$$\mathcal{S}^{-1}(\mathcal{H}) := \{x \in X \mid \mathcal{S}(x) \cap \mathcal{H} \neq \emptyset\} \tag{1.22}$$

of initial states $x \in X$ *from which starts* ***at least one evolution*** $x(\cdot) \in \mathcal{S}(x)$ *satisfying the property* $\mathcal{H}$ *(it is the* inverse image *of* $\mathcal{H}$ *under* $\mathcal{S}$*.).*

For instance, taking for set $\mathcal{H} := \mathcal{X}$ the set of stationary evolutions, we obtain the set of all equilibria x of the evolutionary system: at least one evolution $x(\cdot) \in \mathcal{S}(x)$ remains constant and equal to x. In the same way, taking for set $\mathcal{H} := \mathcal{P}_T(X)$ the set of T-periodic evolutions, we obtain the set of points through which passes at least one T-periodic evolution of the evolutionary system.

When we take $\mathcal{H} := \mathcal{V}(K, C)$ to be the set of evolutions viable in a constrained subset $K \subset X$ outside a target $C \subset K$ (see 1.5, p.16), we obtain the *viability kernel* $\mathrm{Viab}_{\mathcal{S}}(K, C)$ of K outside C:

Definition 1.11.2. [***Viability Kernel and Capture Basin***] *Let $K \subset X$ be a environment and $C \subset K$ be a target.*

1. *The subset* $\mathrm{Viab}_{\mathcal{S}}(K, C)$ *of initial states $x_0 \in K$ such that* ***at least one evolution*** *$x(\cdot) \in \mathcal{S}(x_0)$ starting at x_0 is viable in K for all $t \geq 0$ or viable in K until it reaches C in finite time is called the* viability kernel of K with target C *under $\mathcal{S}$.*

   When the target $C = \emptyset$ is the empty set, we say that $\mathrm{Viab}_{\mathcal{S}}(K) := \mathrm{Viab}_{\mathcal{S}}(K, \emptyset)$ *is the* viability kernel *of K.*

2. *The subset* $\mathrm{Capt}_{\mathcal{S}}(K, C)$ *of initial states $x_0 \in K$ such that* ***at least one evolution*** *$x(\cdot) \in \mathcal{S}(x_0)$ starting at x_0 is viable in K until it reaches C in finite time is called the* capture basin of C viable in K *under $\mathcal{S}$. When $K = X$ is the whole space, we say that* $\mathrm{Capt}_{\mathcal{S}}(C) := \mathrm{Capt}_{\mathcal{S}}(X, C)$ *is the* capture basin *of C.*

We say that

1. a subset K is *viable* under $\mathcal{S}$ if $K = \mathrm{Viab}_{\mathcal{S}}(K)$,
2. K is *viable outside the target* $C \subset K$ under the evolutionary system $\mathcal{S}$ if $K = \mathrm{Viab}_{\mathcal{S}}(K, C)$,
3. C is *isolated* in K if $C = \mathrm{Viab}_{\mathcal{S}}(K, C)$,
4. K is a *repeller* repeller if $\mathrm{Viab}_{\mathcal{S}}(K) = \emptyset$, i.e., if the empty set is isolated in K.

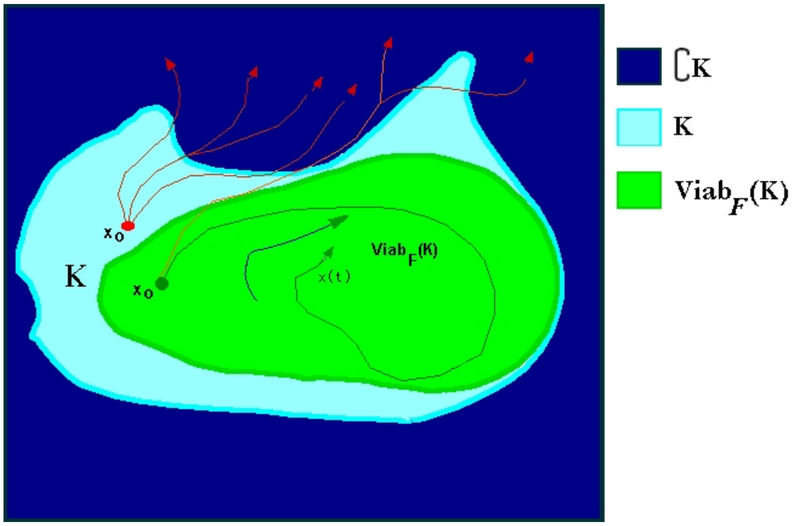


Figure 1.7: [Schema of a Viability Kernel.]

From a point x_0 in the viability kernel starts of the environment K ***at least*** *one evolution viable in K forever.* ***All*** *evolutions starting from $x_0 \in K$ outside the viability kernel leave K in finite time.*

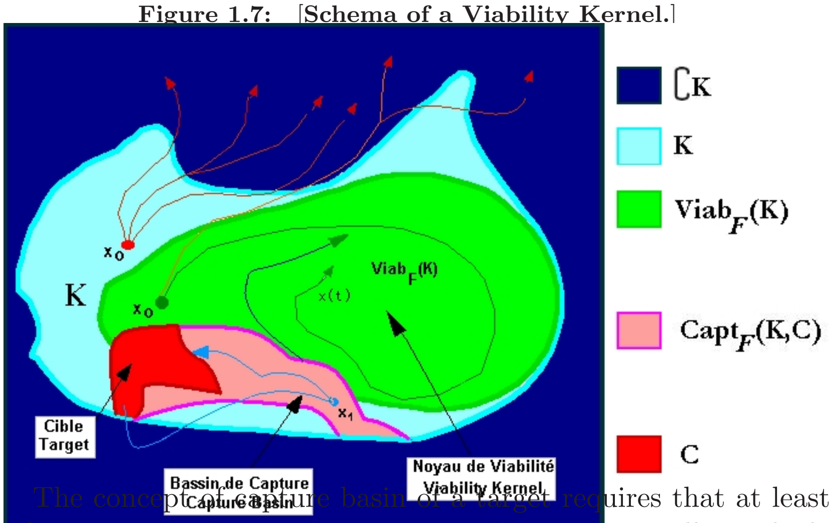


Figure 1.8: [Schema of a Capture Basin.]

From a state x_1 in the capture basin of the target C viable in the environment K starts ***at least*** *one evolution viable in K until it reaches C in finite time.* ***All*** *evolutions starting from $x_2 \in K$ outside the capture basin remain outside the target C until they leave K.*

The concept of capture basin of a target requires that at least en evolution reaches the target in finite time, and *not only asymptotically*, as it is usually studied with concepts of *attractors* since the pioneering works of *Alexander Lyapunov* going back to 1892.

Lemma 1.11.3. [Comparison between Viability Kernels with Targets and Capture Basins] *The viability kernel of K with target C and the capture basin of C viable in K are related by formula*

$$\mathrm{Viab}_{\mathcal{S}}(K, C) = \mathrm{Viab}_{\mathcal{S}}(K \setminus C) \cup \mathrm{Capt}_{\mathcal{S}}(K, C) \tag{1.23}$$

Hence the viability kernel with target C coincides with the capture basin of C viable in K if

$\mathrm{Viab}_{\mathcal{S}}(K \backslash C) = \emptyset$,

Proof —

$$\mathrm{Viab}_{\mathcal{S}}(K, C) \setminus \mathrm{Capt}_{\mathcal{S}}(K, C) \ \subset \ \mathrm{Viab}_{\mathcal{S}}(K \setminus C) \tag{1.24}$$

Proof — Take any $x \in \mathrm{Viab}_{\mathcal{S}}(K, C) \setminus \mathrm{Capt}_{\mathcal{S}}(K, C)$. Since $x \in \mathrm{Viab}_{\mathcal{S}}(K, C)$, there exists at least one evolution $x(\cdot) \in \mathcal{S}(x)$ either viable in K forever or reaching C in finite time. But since $x \notin \mathrm{Capt}_{\mathcal{S}}(K, C)$, all evolutions starting from x are viable in $\complement C$ forever or until they leave K in finite time. Hence the evolution $x(\cdot)$ cannot reach C in finite time, and thus, is viable in K, hence cannot leave K in finite time, and thus is viable in $\complement C$, and consequently, in $K \setminus C$. ■

1.11.2 Viability Kernels under the Lorenz System

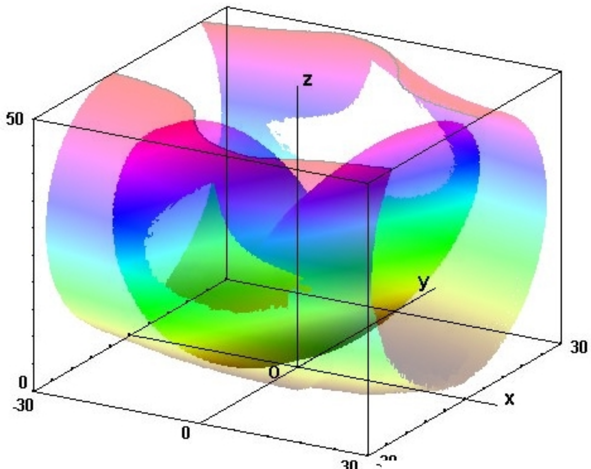
Usually, the attractor, defined as the union of limit sets of evolutions, is approximated by taking the union of the "tails of the trajectories" of the solutions that provides an idea of the shape of the attractor, *although it is not the attractor*. Here, we use the viability kernel algorithm for computing the backward viability kernel, *which contains the attractor*.

Let us consider the Lorenz system

$$\begin{cases} (i) & x'(t) = \sigma y(t) - \sigma x(t) \\ (ii) & y'(t) = r x(t) - y(t) - x(t) z(t) \\ (iii) & z'(t) = \ x(t) y(t) - b z(t) \end{cases}$$

We provide the viability kernel of the cube $[-\alpha, +\alpha] \times [-\beta, +\beta] \times [-\gamma, +\gamma]$ under the Lorenz system (1.6), p.21 and the backward Lorenz system

$$\begin{cases} (i) & x'(t) = -\sigma y(t) + \sigma x(t) \\ (ii) & y'(t) = -r x(t) + y(t) + x(t) z(t) \\ (iii) & z'(t) = \ -x(t) y(t) + b z(t) \end{cases}$$



We take $\sigma > b + 1$, *One can prove that whenever the viability kernel of the backward system is contained in the interior of* K, *the backward viability kernel is contained in the forward viability kernel and that the famous Lorenz attractor is contained in the backward viability kernel. The color scale provides the values of the third coordinates.*

Figure 1.9: [Viability Kernels o
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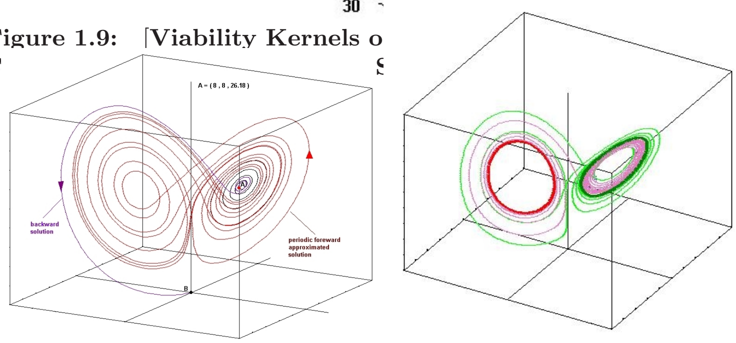


Left: *A periodic evolution is necessarily contained in the limit set, and thus, in the backward viability kernel.* **Right:** *Examples of evolutions viable in the backward viability kernel, which converge to the limit cycles surrounding the nontrivial equilibria.*

Figure 1.10: [Examples of Evolutions Viable in the Backward Viability Kernel]

1.11.3 Perennial Basins

However, reaching the target in finite time is not the end of the story: What does happen next ? If the target C is viable, then the evolution may stay in C forever, whereas it has to leave the target in finite time if the target is a repeller. The capture basin provides the set of initial states from which starts at least one solution viable in K until it reaches the target C in finite time. The behavior of the evolution after it reaches the target C is described by the capture basin $\mathrm{Capt}_{\mathcal{S}}(K, \mathrm{Viab}_{\mathcal{S}}(C))$ of the viability kernel of C viable in K under the evolutionary system $\mathcal{S}$:

Definition 1.11.4. [**Perennial Basin of a Set**] *The* perennial basin *of a nonempty target* C *viable in* K *under the evolutionary system* $\mathcal{S}$ *is the subset of initial states* $x \in K$ *from which starts at least one evolution such that there exists a finite time* $t^\star \geq 0$ *such that*

$$\begin{cases} x(t) \in K & \text{if} \quad t \in [0, t^\star] \\ x(t) \in C & \text{if} \quad t \ \geq \ t^\star \end{cases}$$

In other words, it is the set of initial states from which starts at least one evolution is viable in

K and eventually viable in C forever.

Lemma 1.11.5. [**Viability Characterization of Perennial Basins**] *The perennial basin is equal to the capture basin* $\mathrm{Capt}_{\mathcal{S}}(K, \mathrm{Viab}_{\mathcal{S}}(C))$ *of the viability kernel of* C *viable in* K *under the evolutionary system* $\mathcal{S}$.

It is obviously contained in the viability kernel of K.

If the viability kernel $\mathrm{Viab}_{\mathcal{S}}(C) \subset \mathrm{Int}(C)$ *is contained in the interior of* C*, then*

$$\mathrm{Capt}_{\mathcal{S}}(K, \mathrm{Viab}_{\mathcal{S}}(C)) = \mathrm{Viab}_{\mathcal{S}}(C)$$

The complement

$$\complement(\mathrm{Capt}_{\mathcal{S}}(K, \mathrm{Viab}_{\mathcal{S}}(C))) = \mathrm{Inv}_{\mathcal{S}}(\mathrm{Abs}_{\mathcal{S}}(\complement C), \complement K)$$

of the perennial basin of C *is the set of initial states* $x \in K$ *from which all evolutions* $x(\cdot) \in \mathcal{S}(x)$ *cannot remain in* C *forever: either they leave* K *in finite time or they are viable in* K *and for all* $t < +\infty$*, there exists* $t^\star \geq t$ *with* $x(t^\star) \notin C$.

Remark: — Therefore, we can divide the ball C into three areas:

1. the subset $\mathrm{Viab}_{\mathcal{S}}(C)$ is the set of initial states from which starts at least one evolution viable in C,
2. the subset $(C \cap \mathrm{Capt}_{\mathcal{S}}(K, \mathrm{Viab}_{\mathcal{S}}(C))) \backslash \mathrm{Viab}_{\mathcal{S}}(C)$ of initial states from which at least one evolution leaves C in finite time before returning to C in finite time and then remaining in C forever (such solutions could be called *spike evolutions*, a terminology motivated by the propagation of the nervous influx),
3. the subset $C \backslash \mathrm{Capt}_{\mathcal{S}}(K, \mathrm{Viab}_{\mathcal{S}}(C))$ of initial states $x \in K$ from which all evolutions $x(\cdot)$ either leave K in finite time or are viable in K but cannot remain in C forever. ■

The following example (called Verhulst-Schaeffer model) models the interaction of a fishing activity $y(t)$ and the development of a fish species $x(t)$:

$$\left\{ \begin{array}{ll} x'(t) & = rx(t)\left(1 - \frac{x(t)}{b}\right) - y(t)x(t) \\ y'(t) & = v(t),\ v(t) \in [-1, 1] \end{array} \right. \tag{1.25}$$

The physical constraints of the problem are as follows. The environment K is made of number of fishes above a certain threshold in order to survive. The target is made of made of pairs (x, y) such the fishing activity y is above the hyperbola $C/(\gamma x - c)$ for providing sufficient profits for the economy. Otherwise, fishing activity is in economic crisis. Viability kernels and perennial basins yield the following partition of the environment into three zones:

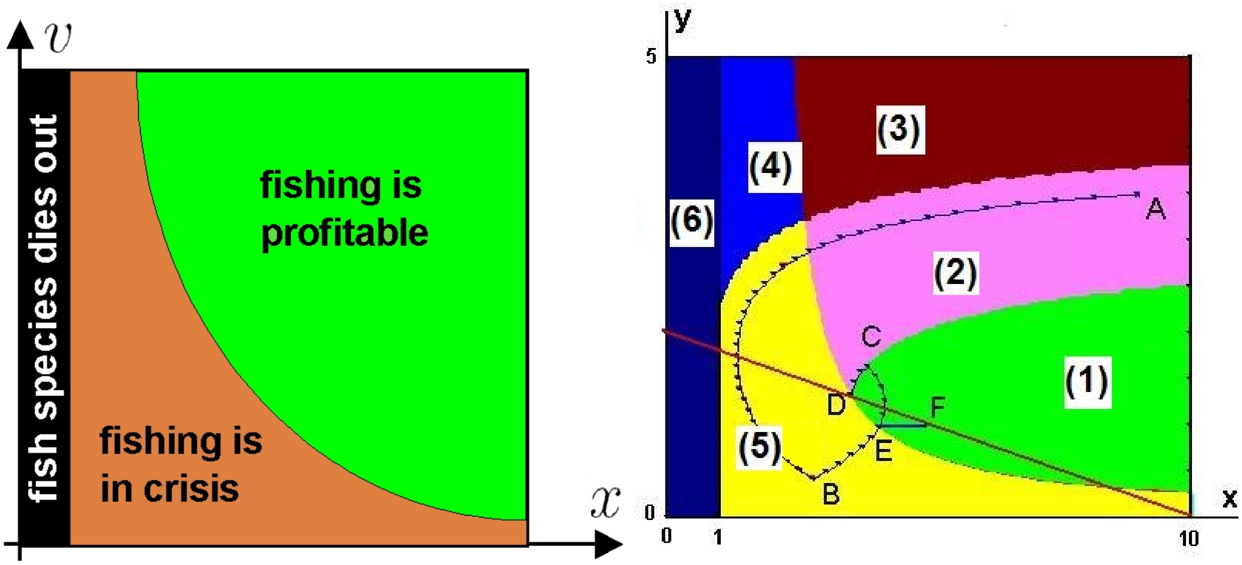


Figure 1.11: [**Perennial Basin of a Target**]
Left: *The subsets K (constraints on the fishes) and C (economic constraints on the fisheries).* ***Right:*** *The perennial basin.*

In other words, the subset C representing the eco(logical&nomical) environment (the root *"eco"* comes from the classical Greek *"oiko"*, meaning house) is partitioned into three zones, Zone 1, the paradise, where economic activity is consistent with the biological viability, Zone 2, the purgatory, where economic activity will eventually disappear but can revive later, and Zone 3, the hell, where economic activity leads to the extinction of fishes.

1.12 The Zermelo Navigation Problem

The Zermelo Navigation Problem *In his 1935 book* Calculus of Variations and partial differential equations of the first order, Constantin Carathéodory *mentions that Zermelo "completely solved by an extraordinary ingenious method" the "Zermelo Navigation Problem" stated as follows:* In an unbounded plane where the wind distribution is given by a vector field as a function of position and time, a ship moves with contant velocity relative to the surrounding air mass. How much the ship be steered in order to come from a starting point to a given goal in the shortest time?

The state variables x and y denote the coordinates of the moving ship, $f(x, y)$ and $g(x, y)$ the components of the wind velocity, and the controls are the *steering direction* u, that is, the angle which the vector of the relative velocity forms with the x-direction, because the components of the absolute velocity are $f(x(t), y(t)) + \cos u(t)$ and $g(x(t), y(t)) + \sin u(t)$, and the norm $\|v\|$ of the velocity $v \in [0, c]$ of the ship. We can also incorporate state-dependent constraints on the steering direction described by the bounds $\alpha(x(t), y(t))$ and $\beta(x(t), y(t))$. We consider a time-independent problem for simplicity.

The evolution of the ship is governed by the control system

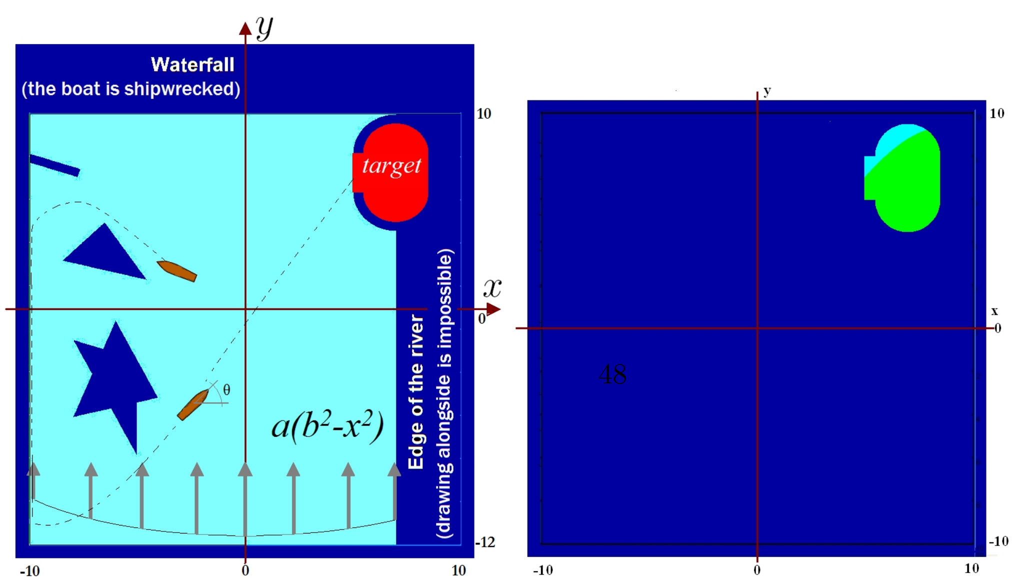
$$\left\{\begin{array}{ll} (i) & x'(t) = f(x(t), y(t)) + v(t)\cos u(t) \\ (ii) & y'(t) = g(x(t), y(t)) + v(t)\sin u(t) \\ & \text{where } u(t) \in [\alpha(x(t), y(t)), \beta(x(t), y(t))]\,; \text{and } v(t) \in [0, c(x(t), y(t))] \\ & 0 \leq \alpha(x, y) \leq \beta(x, y) \leq 2\pi \text{ and } 0 \leq c(x(t, y(t))) \end{array}\right. \tag{1.26}$$

We take up Zermelo's problem, where we replace the "unbounded plane" by an arbitrary closed *environment* $K \subset \mathbb{R}^2$ *with obstacles*, the "goal" being named "target" C in our vocabulary. The target can be regarded itself as a sub-environment, defined by weaker constraints than the original one.

We present these results when the evolution of the ship is governed by Zermelo's equation

$$\left\{\begin{array}{ll} (i) & x'(t) = v(t)\cos u(t) \\ (ii) & y'(t) = a\left(b^2 - x^2\right) + v(t)\sin u(t) \\ & \text{where } u(t) \in [0, 2\pi],\ v(t) \in [0, c] \end{array}\right. \tag{1.27}$$

with $a = \dfrac{1}{80}$, $b = 12$ and $c = 1$.



Letf: *The target C is the harbor, and the environment is union of the rectangle (the sea) deprived of two monstrous obstacles, Skylla and Charybdis and of the harbor.*

Figure 1.12: **[Avoiding Skylla and Charybdis]**

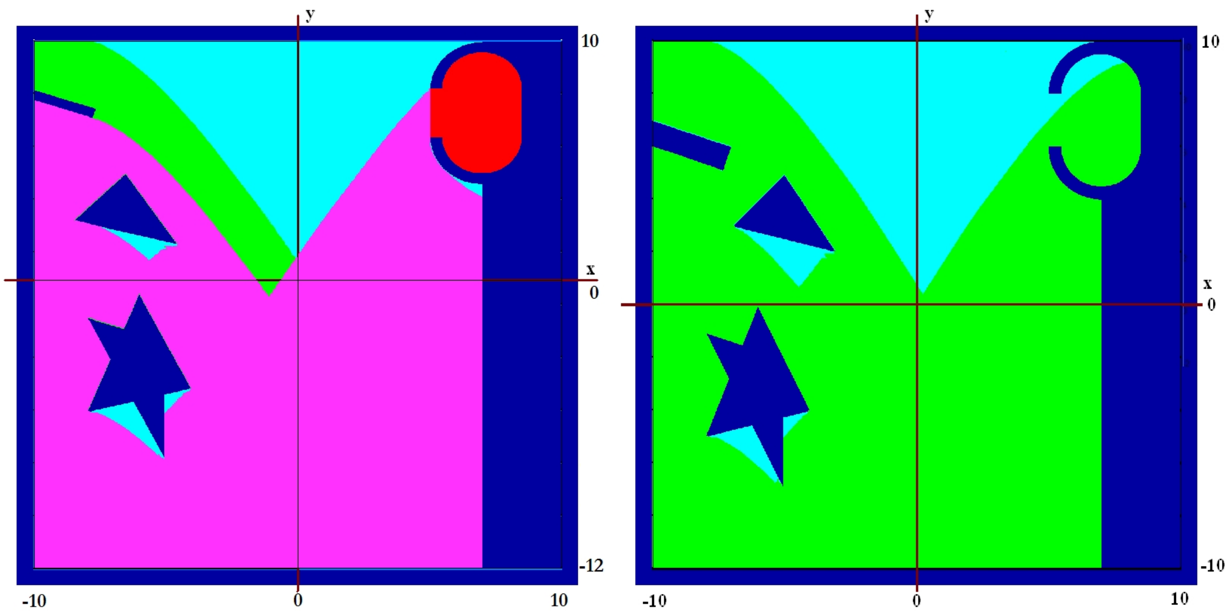


Figure 1.13: [Capture Basin and Perennial Basin of the target]

Left: Capture Basin. Right: Perennial Basin

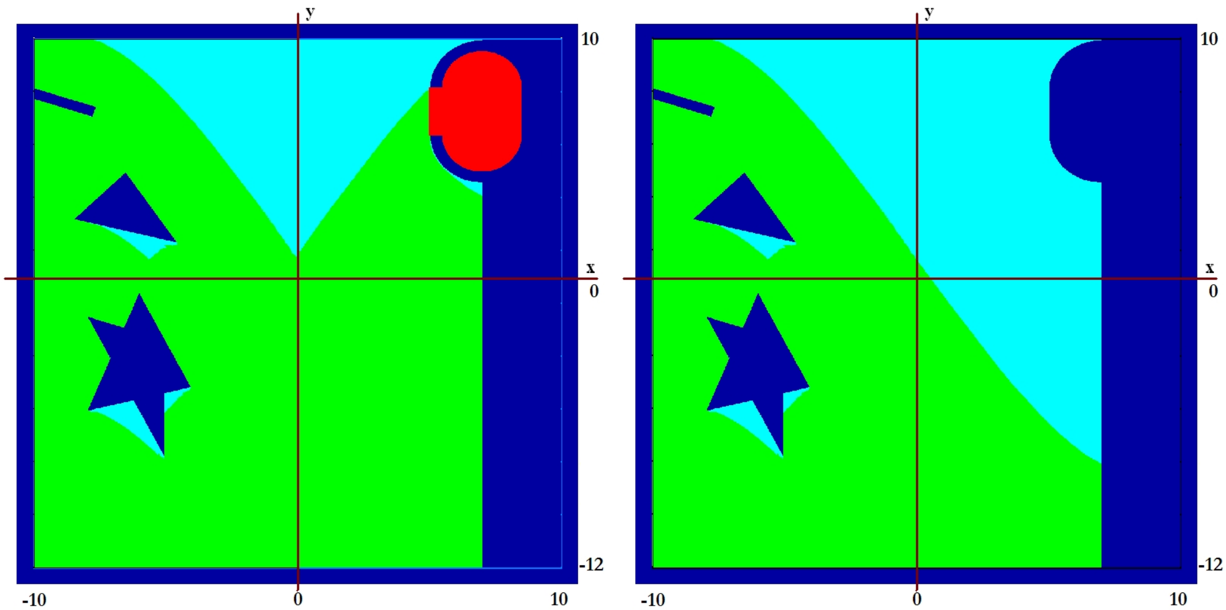


Figure 1.14: [Viability Kernel with Target and Viability Kernel of $K \setminus C$ and Perennial Basin of the target]
Left: Viability Kernel with Target. Right: Viability Kernel of $K \setminus C$

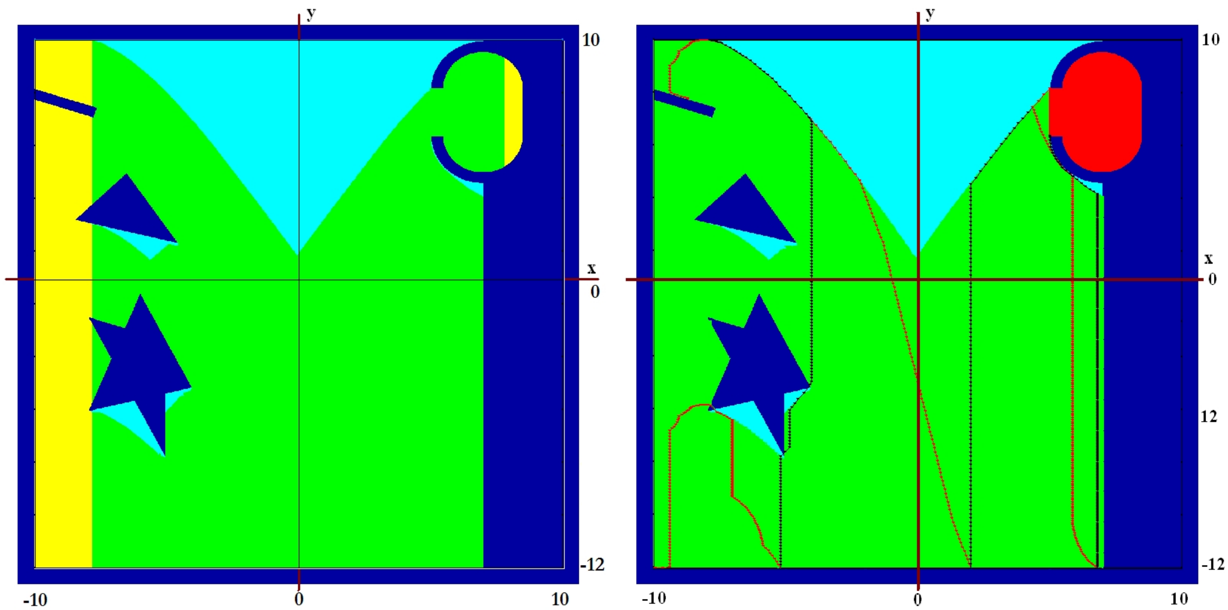


Figure 1.15: [**The Equilibria Set and Examples of trajectories of viable evolutions**]

Left: The set of equilibria. *The Equilibria Set, defined by* $\{x \in K \text{such that } \exists u \in U(x), 0 = f(x,u)\}$. **Right: Arbitrary Evolutions**.

1.13 Exit and Hitting Functions

1.13.1 Epigraphs and Hypograps of Extended Functions

We use the convention $\inf\{\emptyset\} := +\infty$ and $\sup\{\emptyset\} := -\infty$.

Definition 1.13.1. [**Extended Functions and Hidden Constraints**] *A function* $\mathbf{v} : X \mapsto \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ *is said to be an* extended function. *Its* domain $\mathrm{Dom}(\mathbf{v})$ *defined by*

$$\mathrm{Dom}(\mathbf{v}) := \{x \in X \mid -\infty \ < \ \mathbf{v}(x) \ < \ +\infty\}$$

is the set of elements on which the function is finite.

The domain of an extended function incorporates implicitly state constraints hidden in the extended character of the function $\mathbf{v}$.

As in optimization theory, it was discovered since the years 1960 with the development of convex analysis founded by Fenchel, Jean-Jacques Moreau and Rocakfellar that many properties relevant to the optimization of general functionals, involving the order relation of $\mathbb{R}$ and inequalities, are read through their epigraphs or hypographs:

Definition 1.13.2. [**Epigraph of a Function**] *Let* $\mathbf{v} : X \mapsto \overline{\mathbb{R}}$ *be an extended function.*

1. *Its* epigraph $\mathcal{E}p(\mathbf{v})$ *is the set of pairs* $(x, y) \in X \times \mathbb{R}$ *satisfying* $\mathbf{v}(x) \leq y$.
2. *Its* hypograph $\mathcal{H}yp(\mathbf{v})$ *is the set of pairs* $(x, y) \in X \times \mathbb{R}$ *satisfying* $\mathbf{v}(x) \geq y$.

1.13.2 Exit and Hitting Time Functionals and Functions

The concepts of viability kernels and capture basins are closely related to the concepts of exit and hitting time functions.

Definition 1.13.3. [**Exit and Hitting Functionals**] *Let* $K \subset X$ *be a subset.*

1. *The functional* $\tau_K : \mathcal{C}(0, \infty; X) \mapsto \mathbb{R}_+ \cup \{+\infty\}$ *associating with* $x(\cdot)$ *its* exit time $\tau_K(x(\cdot))$ *defined by*
$$\tau_K(x(\cdot)) := \inf \{t \in [0, \infty[\mid x(t) \notin K\}$$
is called the exit functional.
2. *Let* $C \subset K$ *be a target. We introduce the* (constrained) hitting functional *(or* minimal time*)* $\varpi_{(K,C)}$ *defined by*
$$\varpi_{(K,C)}(x(\cdot)) := \inf\{t \geq 0 \mid x(t) \in C \;\&\; \forall s \in [0, t],\ x(s) \in K\ \}$$
associating with $x(\cdot)$ *its hitting time or* minimal time.

These being defined, we apply these functionals to evolutions provided by an evolutionary system:

Definition 1.13.4. [**Upper Exit and Lower Hitting Functions**] *Consider an evolu-*

tionary system $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0, +\infty; X)$. Let $K \subset X$ and $C \subset K$ be two subsets.

1. *The (extended) functions $\tau_K^\flat : K \mapsto \mathbb{R}_+ \cup \{+\infty\}$ and $\tau_K^\sharp : K \mapsto \mathbb{R}_+ \cup \{+\infty\}$ defined by*
$$\tau_K^\flat(x) := \inf_{x(\cdot) \in \mathcal{S}(x)} \tau_K(x(\cdot)) \ \& \ \tau_K^\sharp(x) := \sup_{x(\cdot) \in \mathcal{S}(x)} \tau_K(x(\cdot))$$
are called the lower and upper exit functions *respectively.*

2. *The (extended) functions $\varpi_{(K,C)}^\flat : K \mapsto \mathbb{R}_+ \cup \{+\infty\}$ and $\varpi_{(K,C)}^\sharp : K \mapsto \mathbb{R}_+ \cup \{+\infty\}$ defined by*
$$\varpi_{(K,C)}^\flat(x) := \inf_{x(\cdot) \in \mathcal{S}(x)} \varpi_{(K,C)}(x(\cdot)) \ \& \ \varpi_{(K,C)}^\sharp(x) := \sup_{x(\cdot) \in \mathcal{S}(x)} \varpi_{(K,C)}(x(\cdot))$$
are called the lower and upper (constrained) hitting functions *respectively.*

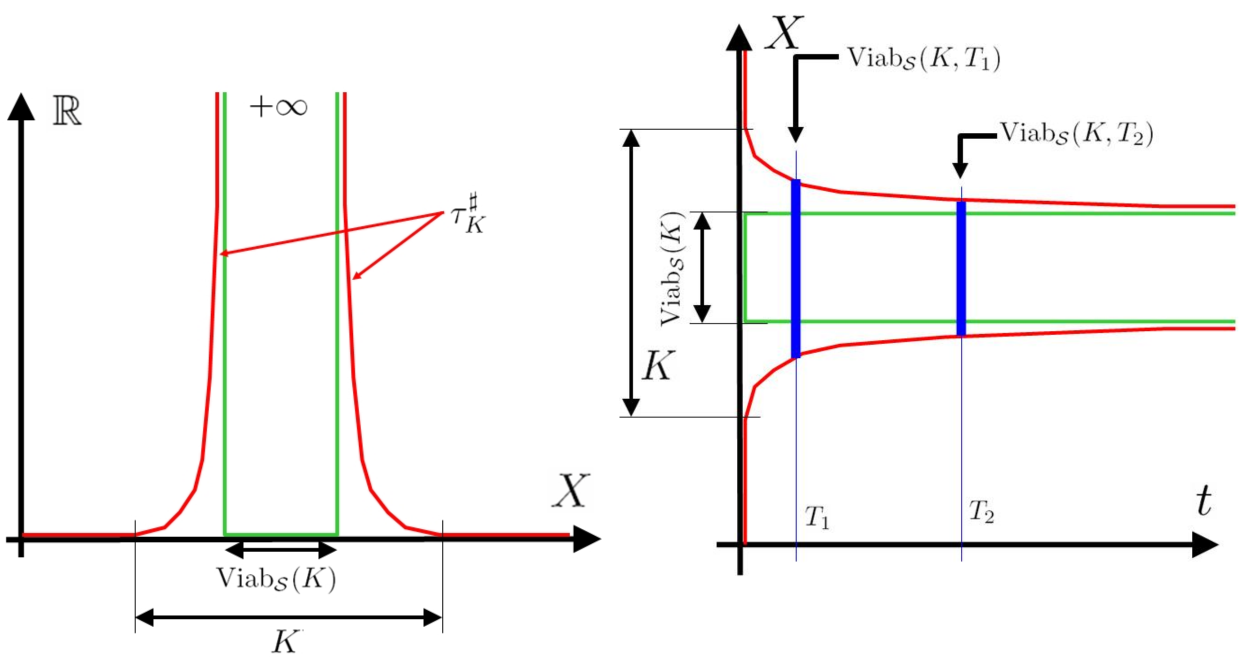


Figure 1.16: [Hypograph of an Upper Exit Function and Graph of the Viability Tube]

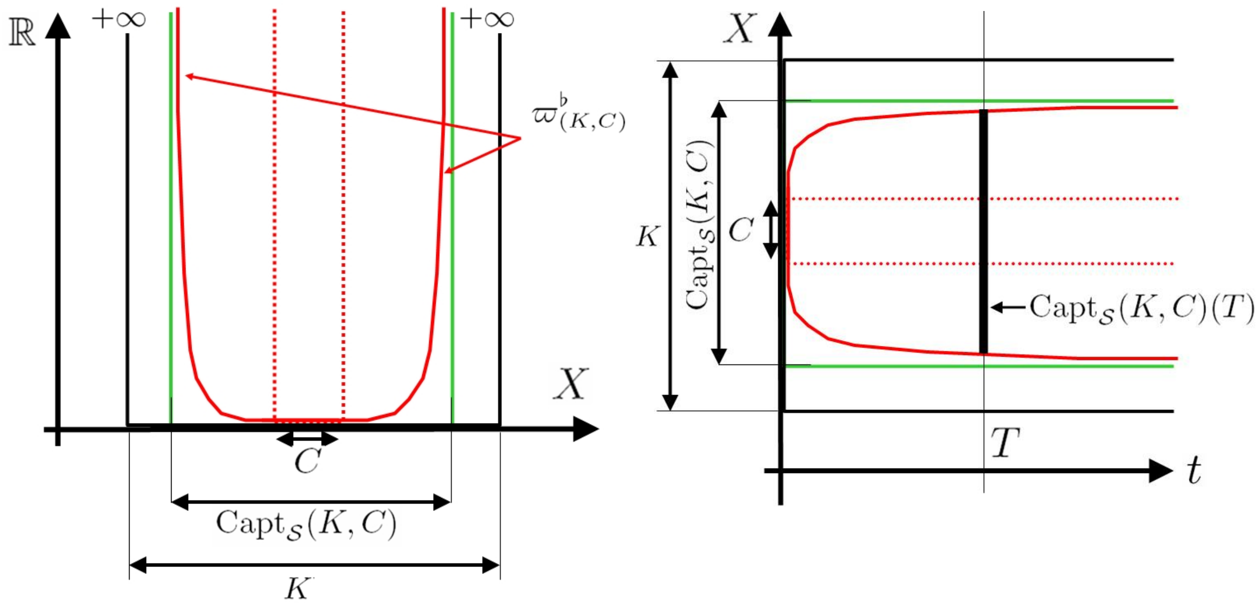


Figure 1.17: [Epigraph of a Lower Hitting Function and Graph of the Capturability Tube.]

We shall relate these functions to viability kernels with targets under the auxiliary system

$$\left\{\begin{array}{ll} (i) & x'(t) = f(x(t), u(t)) \\ (ii) & y'(t) = -1 \\ & \text{where } u(t) \in U(x(t)) \end{array}\right. \tag{1.28}$$

Theorem 1.13.5. [Viability Characterization of Hitting and Exit Functions]

1. *The exit function $\tau^\sharp_K(\cdot)$ is related to the viability kernel by the following formula*

$$\tau^\sharp_K(x) \;=\; \sup_{(x,y)\in \mathrm{Viab}_{(1.28)}(K\times\mathbb{R}_+, K\times\{0\})} y \tag{1.29}$$

2. The hitting function $\varpi^\flat_{K,C}(\cdot)$ is related to the capture basin by the following formula

$$\varpi^\flat_{(K,C)}(x) \; = \inf_{(x,y)\in \mathrm{Viab}_{(1.28)}(K\times\mathbb{R}_+,C\times\mathbb{R}_+)} y \tag{1.30}$$

Proof — Indeed,

1. to say that (x,T) belongs to the viability kernel of $K\times\mathbb{R}_+$ with target $K\times\{0\}$ under evolutionary system (1.28) amounts to saying that there exists an evolution $x(\cdot)\in\mathcal{S}(x)$ starting at x such that $(x(t),T-t)$ is viable in $K\times\mathbb{R}_+$ forever or until it reaches $K\times\{0\}$ at some time $t^\star$. But $T-t$ leaves $\mathbb{R}_+$ at time T and the solution reaches this target at time $t=T$. This means that $x(\cdot)\in\mathcal{S}(x)$ is a solution to the evolutionary system viable in K on the interval $[0,T]$, i.e.,

$$\tau^\sharp_K(x) \; \geq \sup_{(x,T)\in \mathrm{Viab}_{(1.28)}(K\times\mathbb{R}_+,K\times\{0\})} T$$

2. to say that (x,T) belongs to the viability kernel of $K\times\mathbb{R}_+$ with target $C\times\mathbb{R}_+$ under auxiliary system (1.28) amounts to saying that there exists an evolution $x(\cdot)\in\mathcal{S}(x)$ starting at x such that $(x(t),T-t)$ is viable in $K\times\mathbb{R}_+$ for ever or until it reaches $(x(s),T-s)\in C\times\mathbb{R}_+$ at time s. Since $T-s\geq 0$, this means that $x(\cdot)$ is an evolution to the evolutionary system $\mathcal{S}(x)$ viable in K on the interval $[0,s]$ and that $x(s)\in C$, i.e.,

$$\varpi^\flat_{(K,C)}(x) \leq \inf_{(x,T)\in \mathrm{Viab}_{(1.28)}(K\times\mathbb{R}_+,C\times\mathbb{R}_+)} T \; \blacksquare$$

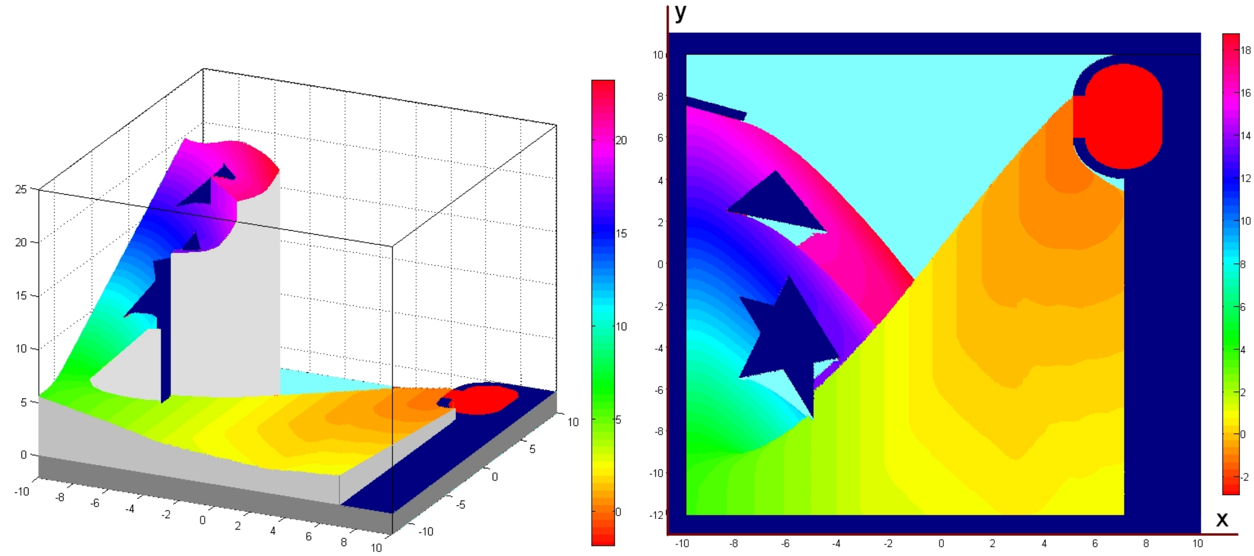


Figure 1.18: [**Minimal Time Function**]
Left: *Capture basin of the target. The capture basin is not connected, made of four pieces, three of them behind the obstacles.* **Right**: *Isochrone Curves (Level Curves of the Minimal Time Function).*

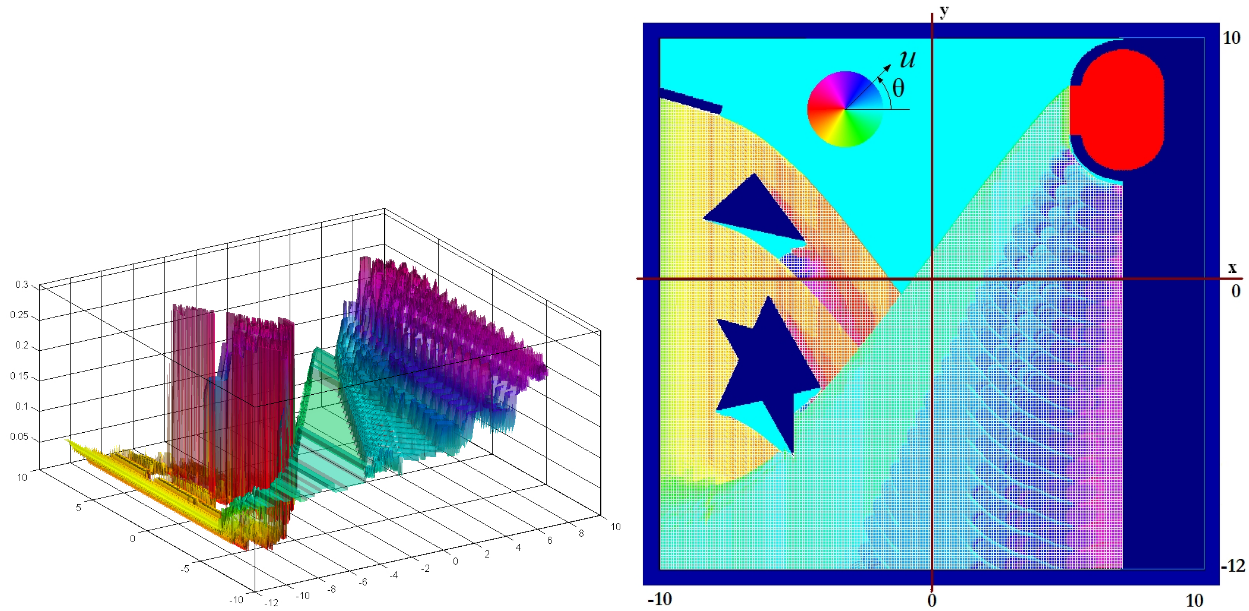


Figure 1.19: [**Minimal Time Function**]

Left: *Graph of the Minimal Time Feedback.* **Right**: *Colored representation of the directions associated with the Minimal Time Feedback.*

Proposition 1.13.6. [***Elementary Properties***]

1. *Behavior under translation: Setting* $(\kappa(-s))(t) := x(t+s)$,

$$\forall s \in [0, \tau_K(x(\cdot))],\ \tau_K((\kappa(-s))(\cdot)) = \tau_K(x(\cdot)) - s \tag{1.31}$$

2. *Monotonicity Properties: If* $K_1 \subset K_2$, *then* $\tau_{K_1}(x(\cdot)) \leq \tau_{K_2}(x(\cdot))$ *and if furthermore,* $C_1 \supset C_2$, *then* $\varpi_{(K_1,C_1)}(x(\cdot)) \leq \varpi_{(K_2,C_2)}(x(\cdot))$

3. *Behavior under union:*

$$\tau_{\bigcup_{i=1}^n K_i}(x(\cdot)) = \min_{i=1,\ldots,n} \tau_{K_i}(x(\cdot))\ \&\ \varpi_{\bigcup_{i=1}^n C_i}(x(\cdot)) = \min_{i=1,\ldots,n} \varpi_{C_i}(x(\cdot))$$

and in particular

$$\forall x \in K \backslash C,\ \tau_{K\backslash C}(x(\cdot)) = \min(\varpi_C(x(\cdot), \tau_K(x(\cdot)))$$

4. Behavior under product:

$$\tau_{\prod_{i=1}^n K_i}(x_1(\cdot), \ldots, x_n(\cdot)) = \min_{i=1,\ldots,n} \tau_{K_i}(x_i(\cdot))$$

Proof — The first two properties being obvious, we note the third holds true since the infimum on a finite union of subsets is the minimum of the infima on each subsets. Therefore, the fourth one follows from

$$\left\{ \begin{array}{l} \tau_{K\backslash C}(x(\cdot)) = \varpi_{\complement(K\backslash C)}(x(\cdot)) = \varpi_{C\cup\complement K}(x(\cdot)) \ = \\ \min(\varpi_C(x(\cdot)), \varpi_{\complement K}(x(\cdot))) = \min(\varpi_C(x(\cdot), \tau_K(x(\cdot)))) \end{array} \right.$$

Observing that when $K := K_1 \times \cdots \times K_n$ where the environments $K_i \subset X_i$ are subsets of vector spaces X_i,

$$\complement\left(\prod_{j=1}^n K_j\right) = \bigcup_{j=1}^n \left(\prod_{i=1}^{j-1} X_i \times \complement K_j \times \prod_{l=j+1}^n X_l\right)$$

the last formula follows from

$$\begin{array}{rcl} \tau_{\prod_{i=1}^n K_i}(x_1(\cdot), \ldots, x_n(\cdot)) & := & \inf\{t \geq 0 \mid x(t) \in \complement K\} \\ & = & \min_{j=1,\ldots,n} \left(\inf\{t | x_j(t) \in \complement K_j\}\right) \\ & = & \min_{j=1,\ldots,n} \tau_{K_j}(x_j(\cdot)) \ \blacksquare \end{array}$$

We introduce the exit subset of K, which is the subset of the boundary through which the evolutions leave K:

Definition 1.13.7. [***Exit Subsets***] *Let us consider an evolutionary system $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0, \infty; X)$ and a subset $K \subset X$. The* exist subset $\mathrm{Exit}_{\mathcal{S}}(K)$ *is the (possibly empty) subset of elements $x \in \partial K$ which leave K immediately:*

$$\mathrm{Exit}_{\mathcal{S}}(K) := \left\{ x \in K \ \ \textit{such that } \tau_K^\sharp(x) = 0 \right\}$$

Exit sets characterize viability and local viability of environments:

Proposition 1.13.8. [***Local Viability Kernel***] *We observe that a subset K is viable under an evolutionary system $\mathcal{S}$ if and only if its exit set* $\text{Exit}_{\mathcal{S}}(K)$ *is empty.*

The subset $K \setminus \text{Exit}_{\mathcal{S}}(K)$ is the largest subset of initial states from which starts at least an evolution locally viable in K.

Furthermore, if K is not viable, the subset K can be covered in the following way:

$$K = \text{Viab}_{\mathcal{S}}(K) \cup \text{Abs}_{\mathcal{S}}(K, \text{Exit}_{\mathcal{S}}(K))$$

Proof — The first statement is obvious, and provides a characterization of viability in terms of exit sets.

If a locally viable evolution $x(\cdot) \in \mathcal{S}(x)$ starts from x, it is clear that $\tau_K^\sharp(x) \geq \tau_K(x(\cdot)) > 0$, so that $x \in K \setminus \text{Exit}_{\mathcal{S}}(K)$. Conversely, if $x \in K \setminus \text{Exit}_{\mathcal{S}}(K))$, i.e., if $\tau_K^\sharp(x) \geq> 0$, then for any $0 < \lambda < \tau_K^\sharp(x)$, there exists $x(\cdot) \in \mathcal{S}(x)$ such that $\tau_K(x(\cdot)) \geq \lambda > 0$, i.e., such that $x(\cdot)$ is viable in K on the nonempty interval $[0, \tau_K(x(\cdot))]$.

The last formulas translates that starting outside the viability kernel of K, all solutions leave K in finite time through the exit set. ■

Proposition 1.13.9. [***Locally Viable Subsets***] *The complement $K \backslash C$ of a target $C \subset K$ in the environment K is locally viable if and only if* $\text{Exit}_{\mathcal{S}}(K) \subset C$.

Proof — Indeed, $K \setminus C$ is locally viable if and only if it is contained in the local viability kernel $K \setminus \text{Exit}_{\mathcal{S}}(K)$, i.e., if and only if $\text{Exit}_{\mathcal{S}}(K) \subset C$. ■

We summarize the semi-continuity properties of the exit and hitting functions in the following statement:

Theorem 1.13.10. [***Semi-Continuity Properties of Exit and Hitting Functions***] *Let us assume that the evolutionary system is upper semicompact and that the subsets K and $C \subset K$ are closed. Then*

1. *the hypograph of the exit function $\tau_K^\sharp(\cdot)$ is closed,*
2. *the epigraph of the hitting function $\varpi^\flat_{(K,C)}(\cdot)$ is closed*

This can be translated by saying that the exit function is upper semicontinuous and the hitting function is lower semicontinuous.

Proof — The first statements follow from Theorems 1.13.5 and 5.3.1. ■

Actually, in several applications, we would like to maximize the exit functional and minimize the hitting or minimal time functional. Indeed, when an initial state $x \in K$ does not belong to the viability kernel, all evolutions $x(\cdot) \in \mathcal{S}(x)$ leave K in finite time. The questions arises to select the "persistent evolutions" in K which persist to remain in K as long as possible:

Definition 1.13.11. [***Persistent Evolutions***] *Let us consider an evolutionary system $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0,\infty;X)$ and a subset $K \subset X$.*

The solutions $x^\sharp(\cdot) \in \mathcal{S}(x)$ which maximize the exit time function

$$\forall x \in K,\ \tau_K(x^\sharp(\cdot)) = \tau_K^\sharp(x) := \max_{x(\cdot)\in\mathcal{S}(x)} \tau_K(x(\cdot)) \tag{1.32}$$

are called persistent evolutions *in K (Naturally, when $x \in \mathrm{Viab}_{\mathcal{S}}(K)$, persistent evolutions starting at x are the viable ones).*
We denote by $\mathcal{S}^{K^\sharp} : K \rightsquigarrow \mathcal{C}(0,\infty;X)$ the evolutionary system $\mathcal{S}^{K^\sharp} \subset \mathcal{S}$ associating with any $x \in K$ the set of persistent evolutions in K.

Definition 1.13.12. [***Minimal Time Evolutions***] *Let us consider an evolutionary system $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0,\infty;X)$ and subsets $K \subset X$ and $C \subset K$.*

The evolutions $x^\flat(\cdot) \in \mathcal{S}(x)$ which minimize the hitting time function

$$\forall x \in K,\ \varpi_{(K,C)}(x^\flat(\cdot)) = \varpi_{(K,C)}^\flat(x) := \min_{x(\cdot)\in\mathcal{S}(x)} \varpi_{(K,C)}(x(\cdot)) \tag{1.33}$$

are called minimal time evolutions *in K.*

Theorem 1.13.13. [***Existence of Persistent and Minimal Time Evolutions***] *Let $K \subset X$ be a closed subset and $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0,\infty;X)$ be an upper semicompact evolutionary system. Then,*

1. *for any $x \notin \mathrm{Viab}_{\mathcal{S}}(K)$, there exists at least one persistent evolution $x^\sharp(\cdot) \in \mathcal{S}^{K^\sharp}(x) \subset \mathcal{S}(x)$ in K,*
2. *for any $x \in \mathrm{Capt}_{\mathcal{S}}(K,C)$, there exists at least one evolution $x^\flat(\cdot) \in \mathcal{S}(x)$ reaching C in minimal time while being viable in K.*

Proof — Let $t < \tau_K^\sharp(x)$ and $n > 0$ such that $t < \tau_K^\sharp(x) - \frac{1}{n}$. Hence there exists an evolution $x_n(\cdot) \in \mathcal{S}(x)$ such that $\tau_K(x_n(\cdot)) \geq \tau_K^\sharp(x) - \frac{1}{n}$, and thus, such that $x_n(t) \in K$. Since the evolutionary system $\mathcal{S}$ is upper semicompact, we can extract a subsequence of evolutions $x_{n'}(\cdot) \in \mathcal{S}(x)$ converging to some evolution $x_\star(\cdot) \in \mathcal{S}(x)$. Therefore, we infer that $x_\star(t)$ belongs to K because K is closed. Since this is true for any $t < \tau_K^\sharp(x)$ and since the evolution $x_\star(\cdot)$ is continuous, we infer that $\tau_K^\sharp(x) \leq \tau_K(x_\star(\cdot))$. Since $\tau_K(x_\star(\cdot)) \leq \tau_K^\sharp(x)$ by definition, we deduce that such an evolution $x_\star(\cdot) \in \mathcal{S}(x)$ is persistent in K.

By definition of $T := \varpi^\flat_{(K,C)}(x)$, for every $\varepsilon > 0$, there exist N such that for $n \geq N$, there exists an evolution $x_n(\cdot) \in \mathcal{S}(x_n)$ and $t_n \leq T_n + \frac{\varepsilon}{2} \leq T + \varepsilon$ such that $x_n(t_n) \in C$ and for every $s < t_n$, $x_n(s) \in K$. Since $\mathcal{S}$ is upper semicompact, a subsequence (again denoted by) $x_n(\cdot)$ converges uniformly on compact intervals to some evolution $x(\cdot) \in \mathcal{S}(x)$. Let us consider also a subsequence (again denoted by) t_n converging to some $T^\star \leq T + \varepsilon$. By passing to the limit, we infer that $x(T^\star)$ belongs to C and that, for any $s < T^\star$, $x(s)$ belongs to K. This implies that

$$\varpi^\flat_{(K,C)}(x) \leq \varpi_{(K,C)}(x(\cdot)) \leq T^\star \leq T + \varepsilon$$

We conclude by letting ε converge to 0: The evolution $x(\cdot)$ obtained above achieves the infimum. ■

Proposition 1.13.14. [***Viability Kernels and Exit Functions***] *Let $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0,+\infty;X)$ be a strict upper semicompact evolutionary system and C and K be two closed subsets such that $C \subset K$. Then the* viability kernel *is characterized by*

$$\mathrm{Viab}_{\mathcal{S}}(K) = \{x \in K \mid \tau_K^\sharp(x) = +\infty\}$$

and the viable-capture basin

$$\mathrm{Capt}_{\mathcal{S}}(K,C) = \{x \in K \mid \varpi^\flat_{(K,C)}(x) \ < \ +\infty\}$$

is the domain of the lower constrained hitting function $\varpi^\flat_{(K,C)}$.

Furthermore, for any $T \geq 0$,

1. *The T-viability kernel $\mathrm{Viab}_{\mathcal{S}}(K)(T)$ of K under $\mathcal{S}$ is equal to the set of elements $x \in K$ from which starts at least one evolution $x(\cdot) \in \mathcal{S}(x)$ viable in K on the interval $[0, T]$,*

2. *the T-viable-capture basins $\mathrm{Capt}_{\mathcal{S}}(K, C)(T)$ of C under $\mathcal{S}$ is equal to the set of elements $x \in K$ from which starts at least one evolution $x(\cdot) \in \mathcal{S}(x)$ viable in K until it reaches the target C before time T.*

Proof — Inclusions

$$\mathrm{Viab}_{\mathcal{S}}(K) \subset \{x \in K \mid \tau_K^\sharp(x) = +\infty\}$$

and

$$\mathrm{Capt}_{\mathcal{S}}(K, C) \subset \{x \in K \mid \varpi_{(K,C)}^\flat(x) < +\infty\}$$

are obviously always true.

Equalities follow from Theorem 1.13.13 by taking the persistent evolutions $x^\sharp(\cdot)$ and minimal time evolutions $x^\flat(\cdot)$ which satisfy the requirements of the theorem. ∎

Theorem 1.13.15. [***Closedness of Exit Sets and Continuity of Exit Functions***] *Let us assume that the evolutionary system is upper semicompact and that the subset K is closed. Then the epigraph of the exit function $\tau_K^\sharp(\cdot)$ is closed if and only if the exit subset $\mathrm{Exit}_{\mathcal{S}}(K)$ is closed, or, equivalently, if and only if the local viability kernel $K \setminus \mathrm{Exit}_{\mathcal{S}}(K)$ is open.*

Proof — Since the closedness of the epigraph of the exit function implies the closedness of the exit subset, let us prove the converse statement. Let us consider a sequence (x_n, y_n) of the epigraph of the exit function converging to some (x, y) and prove that the limit belongs to its epigraph, i.e., that $\tau_K^\sharp(x) \leq y$.

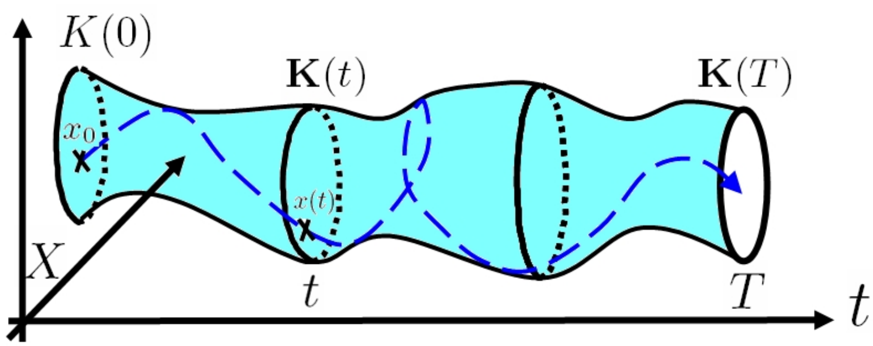
Indeed, since $t_n := \tau_K^\sharp(x_n) \leq y_n \leq y+1$ when n is large enough, there exists a subsequence (again denoted by) t_n converging to $t_\star \leq y$. Since the evolutionary system is assumed to be upper semicompact, there exists a persistent evolution $x_n^\sharp(\cdot) \in \mathcal{S}(x_n)$ such that $t_n := \tau_K(x_n^\sharp(\cdot))$. Furthermore, a subsequence (again denoted by) $x_n^\sharp(\cdot)$ converging to some evolution $x_\star(\cdot) \in \mathcal{S}(x)$ uniformly on the interval $[0, y+1]$. By definition of the persistent evolution, for all $t \in [0, t_n]$, $x_n^\sharp(t) \in K$ and $x_n(t_n) \in \mathrm{Exit}_{\mathcal{S}}(K)$, which is closed by assumption. We thus infer that for all $t \in [0, t_\star]$, $x_\star(t) \in K$ and $x_\star(t_\star) \in \mathrm{Exit}_{\mathcal{S}}(K)$. This means that $t_\star = \tau_K(x_\star(\cdot))$ and consequently, that $\tau_K(x_\star(\cdot)) \leq y$. This completes the proof. ■

Definition 1.13.16. [***Transverse Sets***] *Let $\mathcal{S}$ be an evolutionary system and K be a closed subset. We shall say that K is transverse to $\mathcal{S}$ if for every $x \in K$ and for every evolution $x(\cdot) \in \mathcal{S}(x)$ leaving K in finite time, $\tau_K(x(\cdot)) = \varpi_{\partial K}(x(\cdot))$.*

Transversality of an environment means that all evolutions governed by an evolutionary system cross the boundary as soon as they reach it to leave the environment immediately.

Proposition 1.13.17. [Continuity of the Exit Function of a Transverse Set] *Assume that the evolutionary system $\mathcal{S}$ is upper semicompact and that the subset K is closed and transverse to $\mathcal{S}$. Then the upper exit function $\tau_K^\sharp$ is continuous and the exit set $\mathrm{Exit}_{\mathcal{S}}(K)$ of K is closed.*

1.14 Viability and Capturability Tubes



Tubes are "set-valued evolutions" $\mathbf{K} : t \in \mathbb{R} \rightsquigarrow \mathbf{K}(t) \subset X$.

Figure 1.20: [Tubes]

Definition 1.14.1. [Graph of a Tube] *The graph of the tube $\mathbf{K} : \mathbb{R} \rightsquigarrow X$ is the set of pairs*

(t, x) such that x belongs to $\mathbf{K}(t)$:

$$\text{Graph}(\mathbf{K}) \ = \ \{(t,x) \in \mathbb{R} \times X \textit{ such that } x \in \mathbf{K}(t)\}$$

When $s < T$ are given, we denote by $\mathbf{K}(s \to T)$ the set of evolutions viable in the tube $\mathbf{K}(\cdot)$ *on the time interval $[s, T]$ in the sense that*

$$\forall t \in [s,T], \ x(t) \in \mathbf{K}(t)$$

Definition 1.14.2. [**Viability and Capturability Tubes**] *Let $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0,\infty;X)$ be a evolutionary system and C and K be two closed subsets such that $C \subset K$. The T-viability kernels , the T-capture basins and the T-well are defined by:*

$$\begin{cases} \text{Viab}_{\mathcal{S}}(K)(T) & := \left\{ x \in K \mid \tau_K^{\sharp}(x) \ \geq \ T \right\} \\ \\ \text{Capt}_{\mathcal{S}}(K,C)(T) & :=; \left\{ x \in X \mid \varpi^{\flat}_{(K,C)}(x) \leq T \right\} \\ \\ \text{Well}_{\mathcal{S}}(K,C)(T) & := \{x \in X | \exists x(\cdot) \in \mathcal{S}(x) \textit{ such that } x(T) \in C \& \ \forall t \in [0,T], \ x(t) \in K\} \end{cases} \tag{1.34}$$

*We shall say that the set-valued maps $T \rightsquigarrow \text{Viab}_{\mathcal{S}}(K)(T)$, $T \rightsquigarrow \text{Capt}_{\mathcal{S}}(K,C)(T)$ and $T \rightsquigarrow \text{Well}_{\mathcal{S}}(K,C)(T)$ are respectively the viability tube, the **capturability tube** and and the* well tube*.*

It is clear that $\text{Capt}_{\mathcal{S}}(K,C)(T) \subset \text{Well}_{\mathcal{S}}(K,C)(T)$.

We can characterize the graphs of these tubes:

Proposition 1.14.3. [**The Graph of the Viability and Capturability Tubes**] *Let us consider*

$$\begin{cases} (i) & \tau'(t) = -1 \\ (iI) & x'(t) = f(x(t), u(t)) \\ & \text{where } u(t) \in U(x(t)) \end{cases} \tag{1.35}$$

1. *The graph of the viability tube $\text{Viab}_{\mathcal{S}}(K)(\cdot)$ is the viable-capture basin of $\{0\} \times K$*

viable in $\mathbb{R}_+ \times K$ under the evolutionary system $\mathcal{R}$:

$$\text{Graph}(\text{Viab}_{\mathcal{S}}(K)(\cdot)) = \text{Capt}_{(1.35)}(\mathbb{R}_+ \times K, \{0\} \times K)$$

2. *The graph of the viable-capturability tube* $\text{Capt}(K, C)(\cdot)$ *is the viable-capture basin of $\mathbb{R}_+ \times C$ viable in $\mathbb{R}_+ \times K$ under the evolutionary system $\mathcal{R}$:*

$$\text{Graph}(\text{Capt}_{\mathcal{S}}(K, C)(\cdot)) = \text{Capt}_{(1.35)}(\mathbb{R}_+ \times K, \mathbb{R}_+ \times C)$$

3. *The graph of the Well tube* $\text{Well}(K, C)(\cdot)$ *is the viable-capture basin of $\{0\} \times C$ viable in $\mathbb{R}_+ \times K$ under the evolutionary system $\mathcal{R}$:*

$$\text{Graph}(\text{Well}_{\mathcal{S}}(K, C)(\cdot)) = \text{Capt}_{(1.35)}(\mathbb{R}_+ \times K, \{0\} \times C)$$

Proof — The two first statements follow from Theorem 1.13.5. To say that (T, x) belongs to the viability kernel of $\mathbb{R}_+ \times K$ with target $\{0\} \times C$ under auxiliary system (1.35) amounts to saying that there exists an evolution $x(\cdot) \in \mathcal{S}(x)$ starting at x such that $(x(t), T - t)$ is viable in $K \times \mathbb{R}_+$ for ever or until it reaches $(T - s, x(s)) \in C \times \{0\}$ at time s. Since $T - s = 0$, this means that $x(\cdot)$ is an evolution to the evolutionary system $\mathcal{S}(x)$ viable in K on the interval $[0, T]$ and that $x(T) \in C$, i.e., that x belongs to $\text{Well}_{\mathcal{S}}(K, C)(T)$. ■

Hence the graphs of the viability, capture and well tubes inherit the general properties of capture basins.

1.15 Versatility, inertia and palikinesia

Definition 1.15.1. [**Versatility, Flexibility, Inertia and Palikinesia Functions**] *Let us consider a differentiable evolution $x(\cdot) \in \mathcal{C}(0, \infty; X)$ and a criterion $e : X \mapsto \mathbb{R}_+$ (for example, $e(x) := \|x\|$ or $e(x) := \|x - m\|$).*

We define the versatility *of the evolution with respect to e on the interval $[0, T]$ by*

$$\text{Vers}_e(T; x(\cdot)) := \sup_{t \in [0,T]} e(x'(t))$$

and, when $T = +\infty$,

$$\mathrm{Vers}_e(x(\cdot)) \ := \ \sup_{t \geq 0} e(x'(t))$$

For a differentiable inclusion $x'(t) \in F(x(t))$, the versatility of an evolution $x(\cdot) \in \mathcal{S}_F(x)$ is called the *flexibility*, and the minimal flexibility over viable evolutions

$$\varphi(x) \ := \ \inf_{x(\cdot) \in \mathcal{S}^K(x)} \sup_{t \in [0,T]} e(x'(t))$$

is called the *flexibility function.* For a control system, the versatility of the control is called the *inertia*, and the minimal inertia over the set $\mathcal{P}(x,u)$ of viable solutions $(x(\cdot), u(\cdot))$ to the above parameterized system (1.10) starting at (x,u)

$$\alpha(x,u) \ := \ \inf_{x(\cdot) \in \mathcal{P}(x,u)} \sup_{t \in [0,T]} e(u'(t))$$

is called the *inertia function.* For a tychastic system, the versatility of the tyche is called the *palikinesia* and the minimal palikinesia over viable evolutions

$$\beta(x,v) \ := \ \inf_{x(\cdot) \in \mathcal{P}(x,v)} \sup_{t \in [0,T]} e(v'(t))$$

is called the *palikinesia function.*

The concept of versatility is the counterpart of the *volatility* of the evolution with respect to e defined by

$$\mathrm{Vol}_e(x(\cdot)) \ := \ \left(\int_0^T e(x'(t))^2 dt \right)^{\frac{1}{2}}$$

For example, the *arithmetical* versatility and volatility are

$$\begin{cases} (i) & \mathrm{Vers}_a(x(\cdot)) \ := \ \sup_{t \in [0,T]} \|x'(t) - m\| \\ (ii) & \mathrm{Vol}_a(x(\cdot)) \ := \ \left(\int_0^T \|x'(t) - m\|^2 dt \right)^{\frac{1}{2}} \end{cases}$$

and the *geometrical* versatility and volatility

$$\begin{cases} (i) & \mathrm{Vers}_g(x(\cdot)) \ := \ \sup\limits_{t \in [0,T]} \left\| \dfrac{\|x'(t)\|}{\|x(t)\|} - r \right\| \\ (ii) & \mathrm{Vol}_g(x(\cdot)) \ := \ \left(\displaystyle\int_0^T \left\| \dfrac{\|x'(t)\|}{\|x(t)\|} - r \right\|^2 dt \right)^{\frac{1}{2}} \end{cases}$$

The inverse $\dfrac{1}{\mathrm{Vers}_e(x(\cdot))}$ *of the versatility of an evolution subsumes its "time scale"*. For instance, low versatility corresponds to a long time scale and high versatility is associated with a time short scale.

The computation of the versatilities of evolutions allows us to compare their proper time scale, bringing up a type of hierarchy between them.

Let us set

$$V(x,y) \ := \ \{u \in U(x) \ \text{ such that } e(f(x,u) \ \leq \ y\}$$

The flexibility function can be characterized in terms of the viability kernel of the epigraph of the function $X \times \mathbb{R}_+$ under the auxiliary system:

$$\left\{ \begin{array}{ll} (i) & x'(t) = f(x(t),u(t)) \\ (ii) & y'(t) = 0 \\ & \text{where } u(t) \in V(x(t),y(t)) \end{array} \right. \tag{1.36}$$

subject to the constraint

$$\forall t \geq 0, \ \ (x(t),y(t)) \in K \times \mathbb{R}_+$$

Proposition 1.15.2. [**Viability Characterization of the Flexibility Function**] *The flexibility function is related to the viability kernel of* $K \times \mathbb{R}_+$ *under auxiliary system (1.36) by the following formula*

$$\varphi(x) = \inf_{(x,y) \in \mathrm{Viab}_{(1.36)}(K \times \mathbb{R}_+)} y$$

Proof — Indeed, to say that (x,y) belongs to the viability kernel of $K \times \mathbb{R}_+$ under auxiliary system (1.36) amounts to saying that there exists an evolution $t \mapsto (x(t),y(t))$ governed by the auxiliary system such that, for all $t \geq 0$, $u(t) \in V(x(t),y(t))$. By definition of (1.36), means that $x(\cdot)$ is a solution to the parameterized system, and that for all $t \geq 0$, $y(t) = y$ and

$$e(x'(t)) \ = \ e(f(x(t),u(t))) \ \leq \ y(t) \ = \ y$$

Therefore

$$\sup_{t \geq 0} e(x'(t)) \leq y$$

and thus, $\varphi(e) \leq \inf_{(x,y) \in \mathrm{Viab}_{(1.36)}(K \times \mathbb{R}_+} y$. ■

Hence, the flexibility function inherits the properties of the viability kernels.

Chapter 2

Viability Problems in Management of Renewable Resources

2.1 Example: Evolution of the Biomass of a Renewable Resource

We illustrate some of the basic concepts of viability theory with the study of the evolution of the biomass of one population (of renewable resources, such as fishes in fisheries) in the framework of simple one-dimensional regulated systems. The mention of biomass is just used to provide some intuition to the mathematical concepts and results, but not the other way around, as a "model" of management of renewable resources.

2.1.1 From Malthus to Verhulst and Beyond

We assume that there is a constant supply of resources, no predators and limited space: at each instant $t \geq 0$, the biomass $x(t)$ of the population must remain confined in an interval $K := [a, b]$ describing the *environment* (where $0 < a < b$). The maximal size b that the biomass can achieve is called the *carrying capacity* in the specialized literature.

The dynamics governing the evolution of the biomass are unknown, really. However, several models have been proposed. They are all particular cases of a general dynamical systems of the form

$$x'(t) = \widetilde{u}(x(t))x(t) \tag{2.1}$$

where $\widetilde{u} : [a, b] \mapsto \mathbb{R}$ is a mathematical translation of the *growth rate* of the biomass of the population *feeding back* on the biomass (the specialists of these fields prefer to study growth rates than velocities, as in mechanics or physics). Such a map $\widetilde{u}$ is usually called a *feedback* (also called "*retroaction, closed-loop control*" in control theory).

The scarcity of resources sets a limit to population growth. The question soon arose to know whether the environment $K := [a, b]$ is viable under differential equation (2.1) associated with such or such feedback $\widetilde{u}$ proposed by specialists in population dynamics.

Another question, which we answer in this chapter, is in some sense "inverse": Given an environment, the viability property and maybe other properties required on the evolutions, *what are all the feedbacks $\widetilde{u}$ under which these properties are satisfied*? Answering the second question automatically answers the first one.

1. Thomas Malthus was the first one to address this viability problem and came up with a negative answer. He advocated in 1798 to choose a constant positive growth rate $\widetilde{u}_0(x) = r > 0$, leading to an exponential evolution $x(t) = xe^{rt}$ starting at x. It *leaves the interval* $[a, b]$ *at finite time* $t^\star := \frac{1}{r}\log\left(\frac{b}{x}\right)$ (see left column of Figure 2.1, p.71). In other words, no bounded interval can be viable under Malthusian dynamics. This is the price to pay for linearity of the dynamic of the population: "*Population, when unchecked, increases in a geometrical ratio*", as he concluded in his celebrated *An essay on the principle of population* (1798). He thus was worried by the great poverty of his time, so that he finally recommended "moral restraint" to stimulate savings, diminish poverty, maintain wages above the minimum necessary, and catalyze happiness and prosperity.

   For overcoming this pessimistic conclusions, other explicit feedbacks have next been offered for providing evolutions growing fast when the population is small and declining when it becomes large to compensate for the never ending expansion of the Malthusian model.

2. The Belgium mathematician Pierre-François Verhulst proposed in 1838 the *Verhulst feedback* of the form
   $$\widetilde{u}_1(x) := r(b - x) \text{ where } r > 0$$
   after he had read Thomas Malthus' Essay . It was rediscovered in 1920 by Raymond Pearl and again in 1925 by A. J. Lotka who called it the *law of population growth*.

   The environment K is viable under the associated *purely logistic Verhulst equation* $x'(t) = rx(t)(b - x(t))$. The solution starting from $x \in [a, b]$ is equal to the "sigmoid" $x(t) = \frac{bx}{x + (b - x)e^{-rt}}$. It has the famous S-shape, remains confined in the interval $[a, b]$ and converges to the carrying capacity b when $t \mapsto +\infty$ (see center column of Figure 2.1, p.71). The logistic model and the S-shape graph of its solution became very popular since the 1920's and stood as the evolutionary model of a large manifold of growths, from the tail of rats to the size of men.

The interval $[0, 1]$ is viable under the the Verhulst logistic differential equation $x'(t) = rx(t)(1 - x(t))$ whereas its viability kernel is a Cantor set for its discrete analogue $x_{n+1} = rx_n(1 - x_n)$ when $r > 4$.

3. However, other examples of explicit feedbacks have been used in population dynamics. For instance, the environment K is viable under the following feedbacks: $\widetilde{u}_2(x) := e^{r(b-x)} - 1$, a continuous analogue of a discrete time model by model proposed by Ricker and May, $\widetilde{u}_3(x) := r(b - x)^{\alpha}$, a continuous analogue of a discrete-time model proposed by Hassel and May, the feedback $\widetilde{u}_4(x) := r\left(\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{b}}\right)$, etc.

These feedbacks provide *increasing* evolutions which reach the upper bound b of the environment asymptotically. The three next feedbacks provide viable evolutions reaching b in finite time:

1. **Inert feedbacks**

$$\widetilde{u}_5(x) \ := r\sqrt{2\log\left(\frac{b}{x}\right)}$$

govern evolutions reaching b in finite time with a vanishing velocity so that the state may remain at b forever.

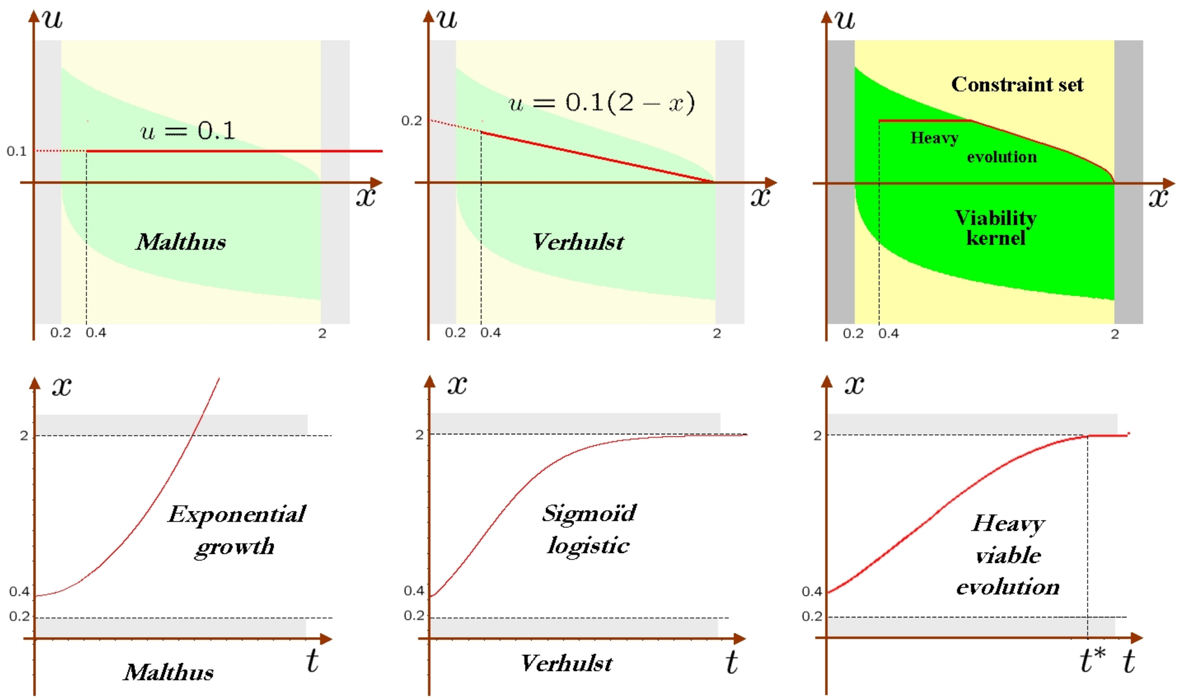


Figure 2.1: [Malthus, Verhulst and Heavy Feedbacks]

2. **Heavy feedbacks** are obtained by "concatenating" the Malthusian and inert feedbacks:

$$\widetilde{u}_6(x) := \begin{cases} r & \text{if} \quad a \leq x \leq be^{-\frac{r^2}{2c}} \\ r\sqrt{2\log\left(\frac{b}{x}\right)} & \text{if} \quad be^{-\frac{r^2}{2c}} \leq x \leq b \end{cases}$$

They govern evolutions combining Malthusian and inert growth: An heavy solution evolves (exponentially) with constant regulon r until the instant when the state reaches $be^{-\frac{r^2}{2c}}$. *This is the last time until which the growth rate could remain constant before being changed by taking*

$$\widetilde{u}(x) = c\sqrt{2\log\left(\frac{b}{x}\right)}$$

Then the evolution follows the inert solution starting and reaches b *at finite time*

$$t^\star := \frac{\log\left(\frac{b}{x}\right)}{r} + \frac{r}{2c}$$

It may remain there forever.

3. **Allee inert feedbacks** are obtained by "concatenating" the following feedbacks:

$$\widetilde{u}_7(x) \ := \left\{ \begin{array}{ll} r\sqrt{2\log\left(\frac{x}{a}\right)} & \text{if} \quad a \leq \ x \ \leq \sqrt{ab} \\ r\sqrt{2\log\left(\frac{b}{x}\right)} & \text{if} \ \ \sqrt{ab} \leq \ x \ \leq \ b \end{array} \right.$$

They govern evolutions combining positive and negative inert growths: An Allee inert evolution evolves with variable regulon $r(t)$ until the instant when the state reaches $\sqrt{ab}$. *This is the last time until which the growth rate could increase before being changed by taking*

$$\widetilde{u}(x) = c\sqrt{2\log\left(\frac{b}{x}\right)}$$

Then the evolution follows the inert solution starting and reaches b *at finite time*. It may remain there forever.

The growth rate feedbacks $\widetilde{u}_i,\ i = 0, \ldots, 7$ are always non negative on the interval $[a, b]$, so that the velocity of the population is always nonnegative, even though the population slows down. Note that $\widetilde{u}_0(b) = r > 0$ is strictly positive at b whereas the values $\widetilde{u}_i(b) = 0,\ i = 1, \ldots, 6$ for all other feedbacks presented above. The growth rates $\widetilde{u}_i,\ i = 0, \ldots, 6$ are not increasing whereas the Allee inert growth rate $\widetilde{u}_7$ is (strictly) increasing on a sub-interval. ■

Instead of finding one feedback $\widetilde{u}$ satisfying the above viability requirements by trial and error, we proceed systematically for designing feedbacks by leaving the choice of the growth rates open, regarding them as *controls* (regulation parameters) of the regulated system system

$$x'(t) = u(t)x(t) \tag{2.2}$$

where the control $u(t)$ is chosen at each time t for governing evolutions confined in the interval $[a, b]$.

We denote by $\mathcal{P}(x, u)$ the set of solutions to system (2.2) viable in the interval $[a, b]$ starting at (x, u). The *inertia function* is defined by

$$\alpha(x, u) := \inf_{x(\cdot)\in\mathcal{P}(x,u)} \sup_{t\geq 0} |u'(t)|$$

The domain $\mathrm{Dom}(\alpha)$ of the inertia function of system $x'(t) = u(t)x(t)$ confronted to environment $K := [a, b]$ is equal to

$$\mathrm{Dom}(\alpha) := (\{a\} \times \mathbb{R}_+) \cup (]a, b[\times \mathbb{R}) \cup (\{b\} \times \mathbb{R}_-)$$

and the inertia function is equal to:

$$\alpha(x,u) \ := \left\{ \begin{array}{ll} \dfrac{u^2}{2\log\left(\frac{b}{x}\right)} & \text{if} \quad a \leq x < b \ \& \ u \geq 0 \\ \dfrac{u^2}{2\log\left(\frac{x}{a}\right)} & \text{if} \quad a < x \leq b \ \& \ u \leq 0 \end{array} \right.$$

The epigraph $\mathcal{E}p(\alpha)$ of the inertia function is closed. However, its domain is neither closed nor open (and not even locally compact). The restriction of the inertia function to its domain is continuous.

Remark: — The inertia function is the unique lower semicontinuous solution (in the generalized sense of Barron-Jensen & Frankowska) to the Hamilton-Jacobi partial differential equation

$$\left\{ \begin{array}{ll} \dfrac{\partial \alpha(x,u)}{\partial x} ux - \alpha(x,u) \dfrac{\partial \alpha(x,u)}{\partial u} = 0 & \text{if} \quad a \leq x < b \ \& \ u \geq 0 \\ \dfrac{\partial \alpha(x,u)}{\partial x} ux + \alpha(x,u) \dfrac{\partial \alpha(x,u)}{\partial u} = 0 & \text{if} \quad a < x \leq b \ \& \ u \leq 0 \end{array} \right.$$

on $\mathrm{Dom}(\alpha)$ with **discontinuous** coefficients. Indeed, the partial derivatives of these two inertia functions are equal to

$$\frac{\partial \alpha(x,u)}{\partial x} \ := \left\{ \begin{array}{ll} \dfrac{u^2}{2x\left(\log\left(\frac{b}{x}\right)^2\right)} & \text{if} \quad u \geq 0 \\ -\dfrac{u^2}{2x\left(\log\left(\frac{x}{a}\right)^2\right)} & \text{if} \quad u \leq 0 \end{array} \right. \quad \& \ \frac{\partial \alpha(x,u)}{\partial u} \ := \left\{ \begin{array}{ll} \dfrac{u}{\log\left(\frac{b}{x}\right)} & \text{if} \quad u \geq 0 \\ \dfrac{u}{\log\left(\frac{x}{a}\right)} & \text{if} \quad u \leq 0 \end{array} \right.$$

Observe that $\frac{\partial \alpha(x,u)}{\partial u}$ is positive when $u > 0$ and negative when $u < 0$. ■

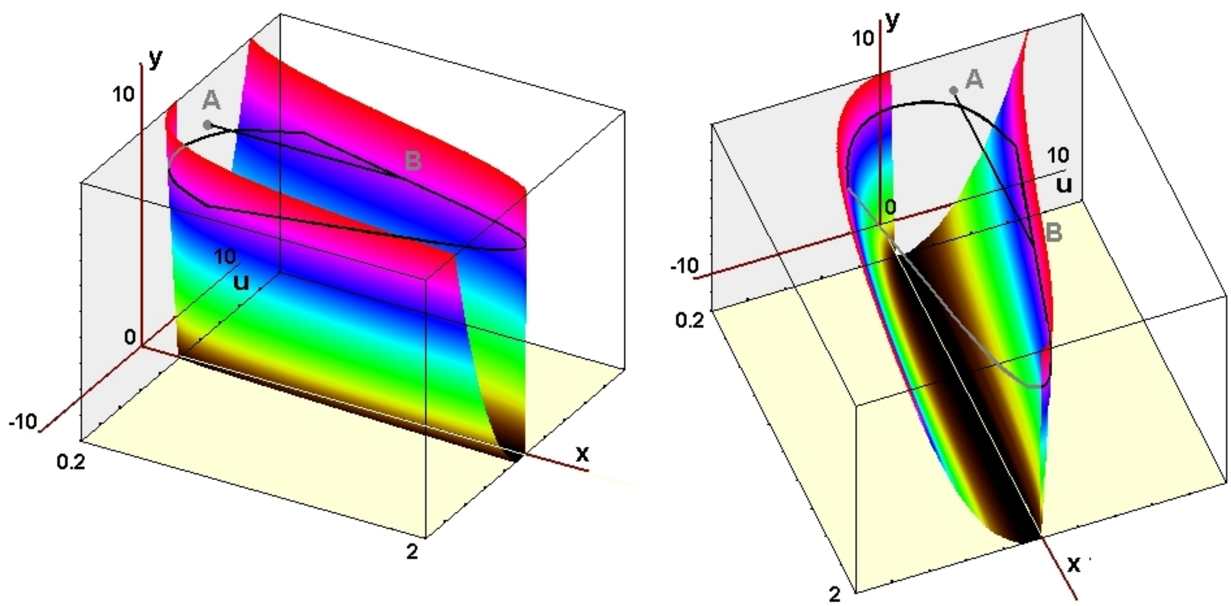


Figure 2.2: [**Inertia Function**]

Two views of the inertia function.

Proposition 2.1.1. *For system $x'(t) = u(t)x(t)$, the* inert regulation map

$$(c,x) \rightsquigarrow R(c;x) := \{u \in \mathbb{R} \ \ \textit{such that}\ \alpha(x,u) \le c\}$$

associated with the inertia function is equal to

$$R(c,x) \ := \begin{cases} \left[0, \sqrt{2c\log\left(\dfrac{b}{a}\right)}\right] & \text{if} \quad x = a \\ \left[-\sqrt{2c\log\left(\dfrac{x}{a}\right)}, \sqrt{2c\log\left(\dfrac{b}{x}\right)}\right] & \text{if} \quad a < x < b \\ \left[-\sqrt{2c\log\left(\dfrac{b}{a}\right)}, 0\right] & \text{if} \quad x = b \end{cases}$$

The critical map $(c,u) \rightsquigarrow \Xi(c;u) := \{x \in [a,b] \ \ \textit{such that}\ \alpha(x,u) = c\}$ *is equal to*

$$\Xi(c,u) \ := \begin{cases} be^{\frac{u^2}{2c}} & \text{if} \quad u > 0 \\ ae^{\frac{u^2}{2c}} & \text{if} \quad u < 0 \end{cases}$$

if $c > 0$ and to

$$\Xi(0,u) := \begin{cases} [a,b] & \text{if} \quad u = 0 \\ \emptyset & \text{if} \quad u \neq 0 \end{cases}$$

if $c = 0$

Theorem 2.1.2. *The inertia function is related to the viability kernel of $\mathcal{K} := [a,b] \times \mathbb{R}_+ \times \mathbb{R}_+$ under the metasystem*

$$\begin{cases} (i) & x'(t) = u(t)x(t) \\ (ii) & u'(t) = v(t) \\ (ii) & y'(t) = 0 \\ & \text{where } |v(t)| \leq y(t) \end{cases} \tag{2.3}$$

by formula

$$\alpha(x,u) = \inf_{(x,u,y) \in \mathrm{Viab}_{(2.3)}(\mathrm{Graph}(U) \times \mathbb{R}_+)} y$$

Proof — Indeed, to say that (x,u,y) belongs to $\mathrm{Viab}_{(\textbf{??})}(\mathrm{Graph}(U) \times \mathbb{R}_+)$ amounts to saying that there exists an evolution $t \mapsto (x(t), u(t))$ governed by (**??**) where $t \mapsto (x(t), u(t), y(t))$ is governed by control system (1.10) and where $y(t) \equiv y$. In other words, the solution $(x(\cdot), u(\cdot)) \in \mathcal{P}(x,u)$ satisfies

$$\forall t \geq 0, \ \|u'(t)\| \leq y$$

so that $\alpha(x,u) \leq \sup_{t \geq 0} \|u'(t)\| \leq y$.

Conversely, if $\alpha(x,u) < +\infty$, we can associate with any $\varepsilon > 0$ an evolution $(x_\varepsilon(\cdot), u_\varepsilon(\cdot)) \in \mathcal{P}(x,u)$ such that

$$\forall t \geq 0, \ \|u'_\varepsilon(t)\| \leq \alpha(x,u) + \varepsilon =: \ y_\varepsilon$$

Therefore, setting $u_{\varepsilon_1}(t) := u'_\varepsilon(t)$ and $y_\varepsilon(t) = y_\varepsilon$, we observe that $t \mapsto (x_\varepsilon(t), u_\varepsilon(t), y_\varepsilon)$ is a solution to the auxiliary system (**??**) viable in $\mathrm{Graph}(U) \times \mathbb{R}_+$, and thus, that (x, u, y_ε) belongs to $\mathrm{Viab}_{(\textbf{??})}(\mathrm{Graph}(U) \times \mathbb{R}_+)$. Hence

$$\inf_{(x,u,y) \in \mathrm{Viab}_{(\textbf{??})}(\mathrm{Graph}(U) \times \mathbb{R}_+)} y \leq y_\varepsilon := \alpha(x,u) + \varepsilon$$

and it is enough to let ε converge to 0. ■

The Viability Theorem provides the analytical formula of the metaregulation map $(x,u,c) \rightsquigarrow G(x,u,c)$ associating with any metastate (x,u,c) the set $G(x,u,c)$ of metacontrols governing the evolution of evolutions with finite inertia:

1. Case when $\alpha(x,u) < c$. Then

$$G(x,u,c) \; := \begin{cases} [0, \alpha(a,u)] & \text{if} \quad x = a \\ [-\alpha(x,u), +\alpha(x,u)] & \text{if} \quad a < x < b \\ [-\alpha(b,u), 0] & \text{if} \quad x = b \end{cases}$$

2. Case when $\alpha(x,u) = c$. Then

$$G(x,u,c) \; := \begin{cases} -\alpha(x,u) & \text{if} \quad u \geq 0 \; \& \; a \leq x < b \\ \alpha(x,u) & \text{if} \quad u \leq 0 \; \& \; a < x \leq b \end{cases}$$

The minimal selection $g^\circ(x,u,c) \in G(x,u,c)$ is equal to $g^\circ(x,u,c) = 0$ if $\alpha(x,u) < c$ and to

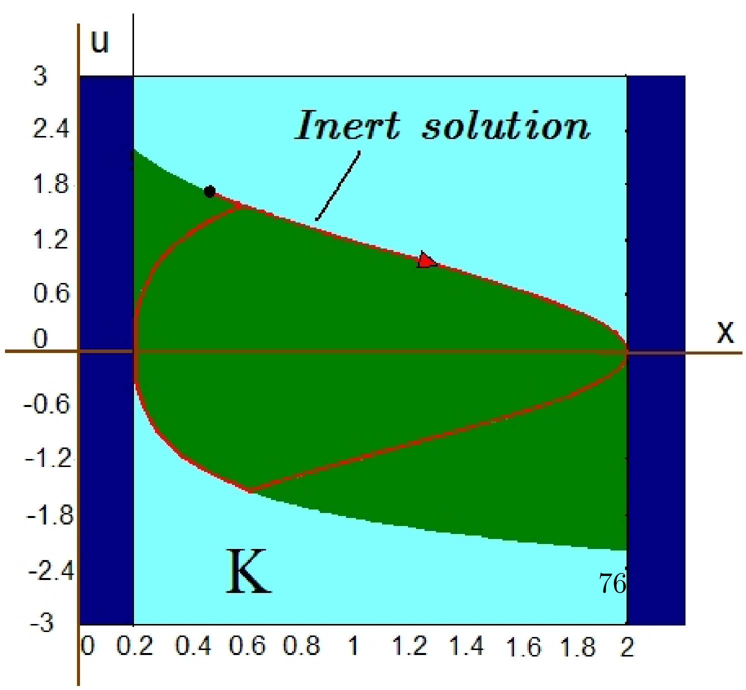
$$g^\circ(x,u,\alpha(x,u)) \; := \begin{cases} -\alpha(x,u) & \text{if} \quad u \geq 0 \; \& \; a \leq x < b \\ \alpha(x,u) & \text{if} \quad u \geq 0 \; \& \; a < x \leq b \end{cases}$$

if $\alpha(x,u) = c$, i.e., if $x \in \Xi(c,u)$ is located in the crisis zone of the control u at inertia threshold c.

Although the minimal selection g° is not continuous, for any initial pair $(x,u) \in \mathrm{Dom}(\alpha)$ in the domain of the inertia function, system of differential equations

$$\begin{cases} (i) & x'(t) = u(t)x(t) \\ (ii) & u'(t) = g^\circ(x(t), u(t), c) \end{cases} \tag{2.4}$$

has solutions which are called *heavy viable evolutions* of initial system (2.2). The trajectory of this heavy evolution is shown on the graph of the inertia function displayed in Figure 2.1.1.



The Saint-Pierre Viability Kernel Algorithm computes the viability kernel (which is the graph of the regulation map U_c) on a sequence of refined grids, provides an arbitrary viable evolution, the heavy evolution minimizing the velocity of the controls and which stops at equilibrium b, and the inert evolutions going back and forth from a to b in an hysteresis cycle.

Figure 2.3: [Viability Kernel and Inert Evolution]

Figure 2.4: [**Heavy evolution**]

The evolution of the growth rate (in blue) of the heavy evolution starting at (x,u) such that $\alpha(x,u) < c$ and $u > 0$ is constant until the time (kairos) $\frac{1}{u}\log\left(\frac{b}{x}\right) - \frac{u}{2c}$ *at which the evolution reaches the crisis zone* $\Xi_c(u) = [be^{-\frac{u^2}{2c}}, b]$. *During this period, the state (in blue) follows an exponential (Malthusian) growth* xe^{ut}. *After, the growth rate decreases linearly until the time* $\frac{1}{u}\log\left(\frac{b}{x}\right) + \frac{u}{2c}$ *when it vanishes and when the evolution reaches the upper bound* b. *During this period, the inertia* $\alpha(x(t),u(t)) = c$ *remains equal to the inertia threshold until the evolution reaches the upper bound* b *with a velocity equal to* 0. *This is an equilibrium at which the evolution may remain forever.*

Remark: The Allee Effect — In order to have negative velocities, we should require that the feedback satisfies the following phenomenological properties: for some $\xi \in [a, b[$,

$$\begin{cases} (i) & \forall x \in [a, \xi[, \ \widetilde{u}'(x) \ > \ 0 \\ (ii) & \forall x \in]\xi, b], \ \widetilde{u}'(x) \ < \ 0 \\ (iii) & u(b) = 0 \end{cases}$$

The increasing behavior of $\widetilde{u}(x)$ on the interval $[a, \xi[$ is called the *Allee effect* (called from Warder Clyde Allee (1885-1955)), stating that at a low population size, an increase of the population size is desirable and has positive effects on population growth, whereas the decreasing behavior of $\widetilde{u}(x)$ on the interval $]\xi, b]$ is called the *logistic effect*, stating that at high population, an increase of the size has a negative effect on the growth of the population.

The heavy feedback has an Allee effect on the interval $[x, \xi_c(u)]$ *and a logistic effect on* $[\xi_c(u), b]$. *Bounding the inertia by c, the feedback governing the heavy evolution maximizes the Allee effect.*

We can obtain feedbacks having a strong Allee effect by concatenating any increasing feedback $\widetilde{w} : [a, \xi_{\widetilde{w},c}] \mapsto \mathbb{R}_+$ with the inert feedback $\sqrt{c}\ r^{\sharp}(x) : [b, \xi_{\widetilde{w},c}] \mapsto \mathbb{R}_+$ where $\xi_{\widetilde{w},c}$ is the root of the equation $\widetilde{w}(\xi) = \xi_{\widetilde{w},c}(\xi)$.

This is the case in particular of the feedback

$$\widetilde{u_7}(x) \ := \begin{cases} \sqrt{c} r^{\flat}(x) & \text{if} \quad a \leq x \leq \sqrt{ab} \\ \sqrt{c}\ r^{\sharp}(x) & \text{if} \quad \sqrt{ab} \leq x \ \leq \ b \end{cases}$$

studied before.

2.1.2 The Inert Hysteresis Cycle

We observe that the graphs of the feedbacks $\sqrt{c}r^\sharp$ and $\sqrt{c}r^\flat$ intersect at the point $(x^\star, \sqrt{c}u^\star)$ where

$$x^\star := \sqrt{ab} \;\&\; u^\star := \sqrt{\log\left(\frac{b}{a}\right)}$$

Therefore, the warning time is equal to

$$\tau^\star = \tau(x^\star, \sqrt{c}u^\star) = 2\frac{\log\left(\frac{b}{x^\star}\right)}{\sqrt{c}u^\star} = \sqrt{\frac{\log\left(\frac{b}{a}\right)}{c}}$$

The inert evolution $(x(\cdot), u(\cdot))$ starting at $(x^\star, \sqrt{c}u^\star)$ is governed by the regulons

$$\forall t \in [0, \tau^\star],\; u(t) = \widetilde{u}_6(x(t)) := \sqrt{c}\, r^\sharp(x(t))$$

During this interval of time, the regulon decreases whereas the biomass continues to increase (the Titanic syndrome due to inertia).

It reaches the metaequilibrium $(b, 0)$ at time $\tau^\star := \tau(x^\star, \sqrt{c}u^\star)$.

At this point, the solution

1. may stop at equilibrium by taking $u(t) \equiv 0$ when $t \geq \tau^\star$,
2. or switch to an evolution governed by the feedback law

$$\forall t \geq \tau^\star,\; u(t) = \widetilde{u}_8(x(t)) := -\sqrt{c}\, r^\sharp(x(t))$$

among (many) other possibilities to find evolutions starting at $(b, 0)$ remaining viable while respecting the velocity limit on the regulons because $(b, 0)$ lies on the boundary of $[a, b] \times \mathbb{R}$.

Using the feedback $\widetilde{u}_8$ for instance, for $t \geq \tau^\star$, we the evolutions $x(t)$ is still defined by

$$\forall t \geq \tau^\star,\; x(t) = xe^{ut - \frac{u^2t^2}{4\log\left(\frac{b}{x}\right)}}$$

and is associated with the regulons

$$\forall t \geq \tau^\star,\; u(t) = u\left(1 - \frac{ut}{2\log\left(\frac{b}{x}\right)}\right)$$

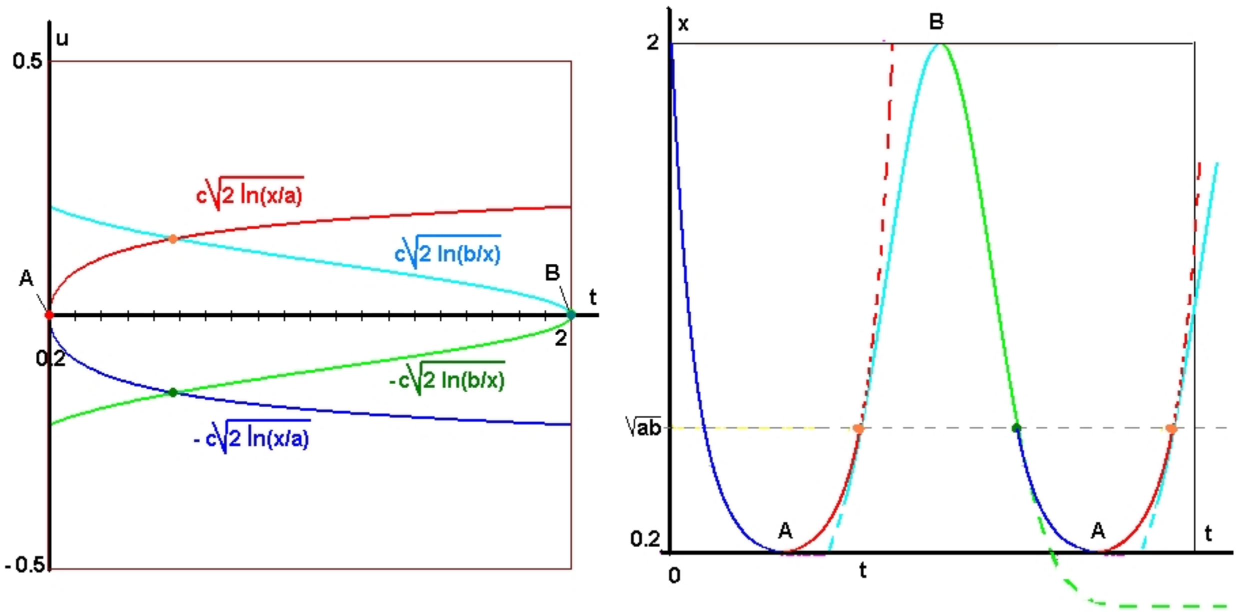


Figure 2.5: [**Graph of the Inert Hysteretic Evolution Computed Analytically**]
This Figure explains how the inert hysteretic evolution can be obtained by piecing together four feedbacks.

Starting from $(x^\star,\sqrt{c}\ u^\star)$ at time 0 with the velocity of the regulon equal to $-c$, the evolution is governed by the inert feedback $\widetilde{u}_6(x) := \sqrt{c}\sqrt{2\log\left(\frac{b}{x}\right)}$ until it reaches b, next governed by the feedback $\widetilde{u}_8(x) := -\sqrt{c}\sqrt{2\log\left(\frac{b}{x}\right)}$ until it reaches $x^\star$, next governed by the inert feedback $\widetilde{u}_9(x) := -\sqrt{c}\sqrt{2\log\left(\frac{x}{a}\right)}$ until it reaches a and last governed by the feedback $\widetilde{u}_{10}(x) := \sqrt{c}\sqrt{2\log\left(\frac{x}{a}\right)}$ until it reaches the point $x^\star$ again. It can be generated automatically by the Viability Kernel Algorithm, so that inert hysteretic evolutions can be computed for non tractable examples. See Figure 2.6.

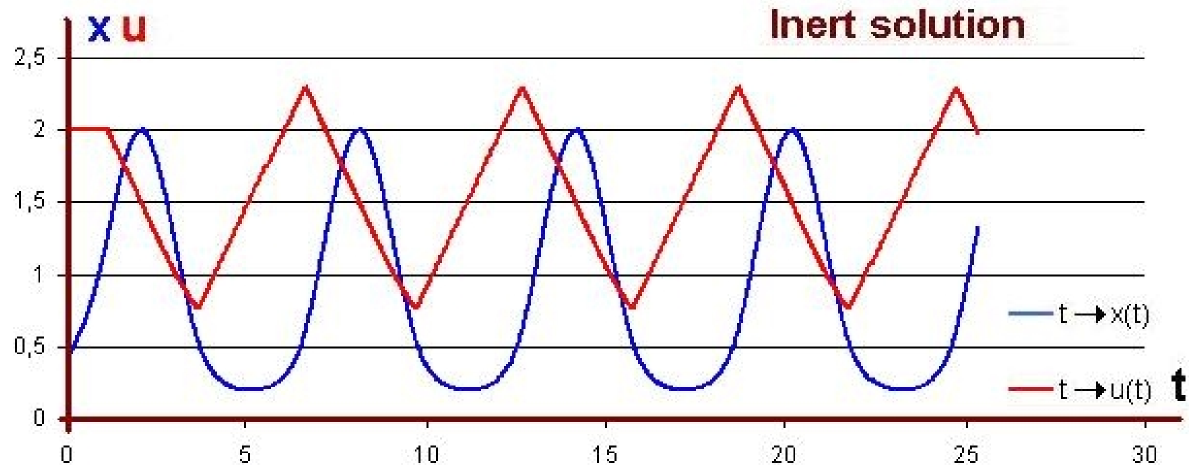


Figure 2.6: [**Graph of the Inert Hysteretic Evolution Computed by the Viability Kernel Algorithm**]
Both the graphs of the inert hysteretic evolution (in blue) and of its control (in red) are plotted. They are not computed from the analytical formulas as in Figure 2.5, but extracted from the Viability Kernel Algorithm.

The metastate $(x(\cdot), u(\cdot))$ is actually governed by the metasystem

$$\begin{cases} (i) & x'(t) = u(t)x(t) \\ (ii) & u'(t) = -c \end{cases}$$

It ranges over the graph of the map $\sqrt{c}\, r^\sharp$ between 0 and $\tau^\star$ and over the graph of the map $-\sqrt{c}\, r^\sharp$ between $\tau^\star$ and $2\tau^\star$. During this interval of time, both the regulon $u(t)$ and the biomass $x(t)$ starts decreasing. The velocity of the negative regulon is constant and still equal to $-\alpha(x, u)$.

But it is no longer viable on the interval $[a, b]$, because with such a strictly negative velocity $-\alpha(x, u)$, $x(\cdot)$ leaves $[a, b]$ in finite time. Hence regulons have to be switched before the evolution leaves the graph of U_c by crossing through the graph of $-\sqrt{c}\, r^\flat$ when $-\sqrt{c}\, r^\flat(x^\star) = -\sqrt{c}\, r^\sharp(x^\star)$ at time $2\tau^\star$.

Therefore, in order to keep the evolution viable,it is the last instant to switch the velocity of the regulon from $-c$ to $+c$.

Starting at $(x^\star, \sqrt{c}u^\star)$ at time $2\tau^\star$, we let the metastate $(x(\cdot), u(\cdot))$ evolve according the metasystem

$$\begin{cases} (i) & x'(t) = u(t)x(t) \\ (ii) & u'(t) = +c \end{cases}$$

It is governed by the regulons

$$\forall t \in [0, \tau^\star],\ u(t) = \widetilde{u}_9(x(t)) := -\sqrt{c}\, r^\flat(x(t))$$

and ranges over the graph of the map $-\sqrt{c}\, r^\flat$ between $2\tau^\star$ and $3\tau^\star$. During this interval of time, the regulon increases whereas the biomass continues to decrease (the Titanic syndrome again due to inertia) and stops when reaching the metaequilibrium $(a, 0)$ at time $3\tau^\star$.

Since $(a, 0)$ lies on the boundary of $[a, b] \times \mathbb{R}$, there are (many) other possibilities to find evolutions starting at $(a, 0)$ remaining viable while respecting the velocity limit on the regulons. Therefore, we continue to use the above metasystem with velocity $+c$ starting at $3\tau^\star$. The evolutions $x(t)$ obtained through the feedback law

$$\forall t \geq \tau^\star,\ u(t) = \widetilde{u}_{10}(x(t)) := +\sqrt{c}\, r^\flat(x(t))$$

The metastate $(x(\cdot), u(\cdot))$ ranges over the graph of the map $\sqrt{c}\, r^\flat$ between $3\tau^\star$ and $4\tau^\star$. During this interval of time, both the regulon $u(t)$ and the biomass $x(t)$ increase until reaching the pair $(x^\star, \sqrt{c} u^\star)$, the initial metastate.

Therefore, letting the heavy solution bypass the equilibrium by keeping its velocity equal to $+c$ instead of switching it to 0, allows us to build a periodic evolution by taking velocities of regulons equal successively to $-c$ and $+c$ on the intervals $[2n\tau^\star, (2n+1)\tau^\star]$ and $[(2n+1)\tau^\star, (2n+2)\tau^\star]$ respectively. We obtain in this way a periodic evolution of period $4\tau^\star$ showing an *hysteresis property*: *The evolution oscillates between a and b back and forth by ranging alternatively two different trajectories on the metaenvironment* $[a, b] \times \mathbb{R}$. The evolution of the state is governed by concatenating four feedbacks, $\widetilde{u}_6 := +\sqrt{c} r^\sharp$ on $[x^\star, b]$, $\widetilde{u}_8 := -\sqrt{c} r^\sharp$ on $[x^\star, b]$, $\widetilde{u}_9 := -\sqrt{c} r^\flat$ on $[a, x^\star]$ and $\widetilde{u}_{10} := +\sqrt{c} r^\flat$ on $[a, x^\star]$.

Note also that not only this evolution is periodic, but obeys a *quantized mode* of regulation: We use only two metacontrols $-c$ and $+c$ to control the metasystem (instead of an infinite family of open loop controls $v(\cdot) := u'(\cdot)$ (as in the control of rockets in space). This is also an other advantage of replacing a control system by its metasystem: use a finite number (quantization) of controls ... to the price of increasing the dimension of the system by replacing it by its metasystem.

We can adapt the inert hysteresis cycle to the heavy case when we start with a given regulon $u < \sqrt{c}\ u^\star = \sqrt{c \log\left(\frac{b}{a}\right)}$. We obtain a periodic evolution by taking velocities of regulons equal successively to 0, $-c$, 0, $+c$, and so on showing an *hysteresis property*: *The evolution oscillates between a and b back and forth by taking two different routes.*

It is described in the following way. We denote by $a_c(u)$ and $b_c(u)$ the roots

$$a_c(u) = ae^{\frac{u^2}{2c}} \;\&\; b_c(u) = be^{-\frac{u^2}{2c}}$$

of the equations $r^\flat(x) = u$ and $r^\sharp(x) = u$ and we set

$$\tau^\star(u) = 2\frac{\log\left(\frac{b}{a}\right)}{u}$$

1. The metastate $(x(\cdot), u(\cdot))$ starts from $(a_c(u), u)$ by taking the velocity of the regulon equal to 0. It remains viable on the time interval $[0, \tau^\star(u) - \frac{u}{2c}]$ until it reaches the metastate $(b_c(u), u)$.

2. The metastate $(x_h(\cdot), u_h(\cdot))$ starts from $(b_c(u), u)$ at time $\tau^\star(u) - \frac{u}{2c}$ by taking the velocity of the regulon equal to $-c$. It is regulated by the metasystem

$$\left\{ \begin{array}{ll} (i) & x'(t) = u(t)x(t) \\ (ii) & u'(t) = -c \end{array} \right.$$

ranging successively over the graphs of $\sqrt{c}\, r^\sharp$ and $-\sqrt{c}\, r^\sharp$ on the time interval $[\tau^\star(u) - \frac{u}{2c}, \tau^\star(u) + \frac{3u}{2c}]$ until it reaches the metastate $(b_c(u), -u)$.

3. The metastate $(x_h(\cdot), u_h(\cdot))$ starts from $(b_c(u), -u)$ at time $\tau^\star(u) + \frac{3u}{2c}$ by taking the velocity of the regulon equal to 0. It remains viable on the time interval $[\tau^\star(u) + \frac{3u}{2c}, 2\tau^\star(u) + \frac{u}{c}]$ until it reaches the metastate $(a_c(u), -u)$.

4. The metastate $(x_h(\cdot), u_h(\cdot))$ starts from $(a_c(u), -u)$ at time $2\tau^\star(u) + \frac{u}{c}$ by taking the velocity of the regulon equal to $+c$. It is regulated by the metasystem

$$\left\{ \begin{array}{ll} (i) & x'(t) = u(t)x(t) \\ (ii) & u'(t) = +c \end{array} \right.$$

ranging successively over the graphs of $-\sqrt{c}\, r^\flat$ and $\sqrt{c}\, r^\flat$ on the time interval $[\tau^\star(u) - \frac{u}{2c}, \tau^\star(u) + \frac{3u}{2c}]$ until it reaches the metastate $(a_c(u), u)$.

In summary, the study of inertia functions and metasystems allowed us to discover several families of feedbacks or concatenation of feedbacks providing several periodic viable evolutions, using two (for the inert hysteresis cycle) or three (for the heavy hysteresis cycle) metacontrols" $+c$, -c and, for the heavy cycle, $+c$, -c and 0.

2.2 Management of Renewable Resources

Let us consider a given non negative growth rate feedback $\widetilde{u}$ governing the evolution of the biomass of a renewable resource $x(t) \geq a > 0$, through differential equation : $x'(t) = \widetilde{u}(x(t))x(t)$. We shall take as examples the Malthusian feedback $u_0(x) := u$, the Verhulst feedback $u_1(x) := r(x-b)$ and the inert feedback $\widetilde{u}_5(x) := r\sqrt{2\log\left(\frac{b}{x}\right)}$.

The evolution is slowed down by industrial activity which depletes it, such as fisheries.

We denote by $v \in \mathbf{R}_+$ the industrial effort for exploiting the renewable resource, playing now the role of the control. Naturally, the industrial effort is subjected to state-dependent constraints $V(x)$ describing economic constraints.

We integrate the ecological constraint by setting $V(x) = \emptyset$ whenever $x < a$.

Hence the evolution of the biomass is regulated by the control system

$$\begin{cases} (i) & x'(t) = x(t)\left(\widetilde{u}(x(t)) - v(t)\right) \\ (ii) & v(t) \in V(x(t)) \end{cases} \tag{2.5}$$

We denote by $\mathcal{Q}_{\widetilde{u}}(x, v)$ the set of solutions to system (2.5). The *inertia function* is defined by

$$\beta_{\widetilde{u}}(x, v) := \inf_{x(\cdot)\in\mathcal{Q}_{\widetilde{u}}(x,v)} \sup_{t\geq 0} |u'(t)|$$

This nonnegative function can take infinite values: It is a function from Graph(V) to $\mathbb{R}\cup\{+\infty\}$. Such function, called an *extended function*, is characterized by its *epigraph* defined by

$$\mathcal{E}p(\beta_{\widetilde{u}}) := \{(x, v, c) \in \text{Graph}(V) \times \mathbb{R}_+ \ \text{ such that } \beta_{\widetilde{u}}(x, v) \leq c\}$$

and is finite on its *domain* defined by

$$\text{Dom}(\beta_{\widetilde{u}}) := \{(x, v) \in \text{Graph}(V) \ \text{ such that } \beta_{\widetilde{u}}(x, v) \ < \ +\infty\}$$

This function is characterized as the viability kernel of a subset under an auxiliary system, known as its *"metasystem"*. It inherits the properties of the viability kernel of an environment and can be computed by the Viability Kernel Algorithm.

This statement follows from the characterization of the epigraph of the inertia function as the viability kernel of the *"metaenvironment"* $\mathcal{K} := [a, b] \times \mathbb{R}_+ \times \mathbb{R}_+$ under the metasystem

$$\begin{cases} (i) & x'(t) = (\widetilde{u}(x(t)) - v(t))x(t) \\ (ii) & v'(t) = w(t) \\ (ii) & y'(t) = 0 \\ & \text{where } |w(t)| \leq y(t) \end{cases} \tag{2.6}$$

The metasytem is regulated by the *"metaregulons"* made of the *velocities* of the former regulons. In other words, the metasystem regulates the evolution of the initial system by modifying the former regulons by acting on their velocities. The component y of the *"metastate"* is the *"inertia threshold"* setting an upper bound to the velocities of the regulons. Therefore, metasystem (2.6) governs the evolutions of the state $x(t)$, the control $v(t)$ and the inertia threshold $y(t)$ by imposing constraints on the velocities of the regulons

$$\forall t \geq 0,\ |v'(t)| \leq y(t)$$

called metaregulons and used as auxiliary regulons.

Unfortunately, the metaenvironment $\mathrm{Graph}(V)$ is obviously not viable under the above metasystem: Every solution starting from $(a(u), v, c)$ with $u < v$ leave it immediately.

Theorem 2.2.1. *The epigraph* $\mathcal{E}p(\beta_{\widetilde{u}})$ *of the inertia function* $\beta_{\widetilde{u}}$ *is equal to the viability kernel* $\mathrm{Viab}_{(2.6)}(\mathrm{Graph}(V) \times \mathbb{R}_+)$ *of the metaenvironment* $\mathrm{Graph}(V) \times \mathbb{R}_+$ *under metasystem (2.6).*

We observe that the inertia function vanishes on the equation line:

$$\beta_{\widetilde{u}}(x, \widetilde{u}(x)) = 0$$

It is identically equal to 0 if for any $x \geq a$, $\widetilde{u}(x) \geq \mathbf{v}(x)$ and identically infinite if for any $x \geq a$, $\widetilde{u}(x) < \mathbf{v}(x)$.

The basic economic model was originated by Graham and taken up by Schaeffer. They assumed that the exploitation rate is proportional to the biomass and the economic activity: viability constraints are described by *economic constraints*

$$\forall t \geq 0,\ cv(t) + C\ \leq\ \gamma v(t)\, x(t)$$

where $C \geq 0$ is a fixed cost, $c \geq 0$ the unit cost of economic activity and $\gamma \geq 0$ the price of the resource. We also assume that

$$\forall t \geq 0,\ 0 \leq v(t) \leq \overline{v}$$

where $\overline{v} > \frac{C}{\gamma x - c}$ is the maximal exploitation effort. Hence the Graham-Schaeffer constraints are summarized under the set-valued map $V : [a, \infty[\rightsquigarrow \mathbb{R}_+$ defined by

$$\forall x \geq a,\ V(x)\ \left[\frac{C}{\gamma x - c}, \overline{v}\right]$$

More generally, we assume that there exists a decreasing positive map $\mathbf{v} : [a, b] \mapsto [0, \overline{v}]$ such that

$$\forall x \in [a, \infty[, := V(x) := [\mathbf{v}(x), \overline{v}]$$

In any case, the epigraph of the inertia function being a viability kernel, it can be computed the Saint-Pierre Viability Kernel Algorithm. Figure 2.7 provides the level sets of the inertia function for the the Verhulst and inert feedbacks respectively.

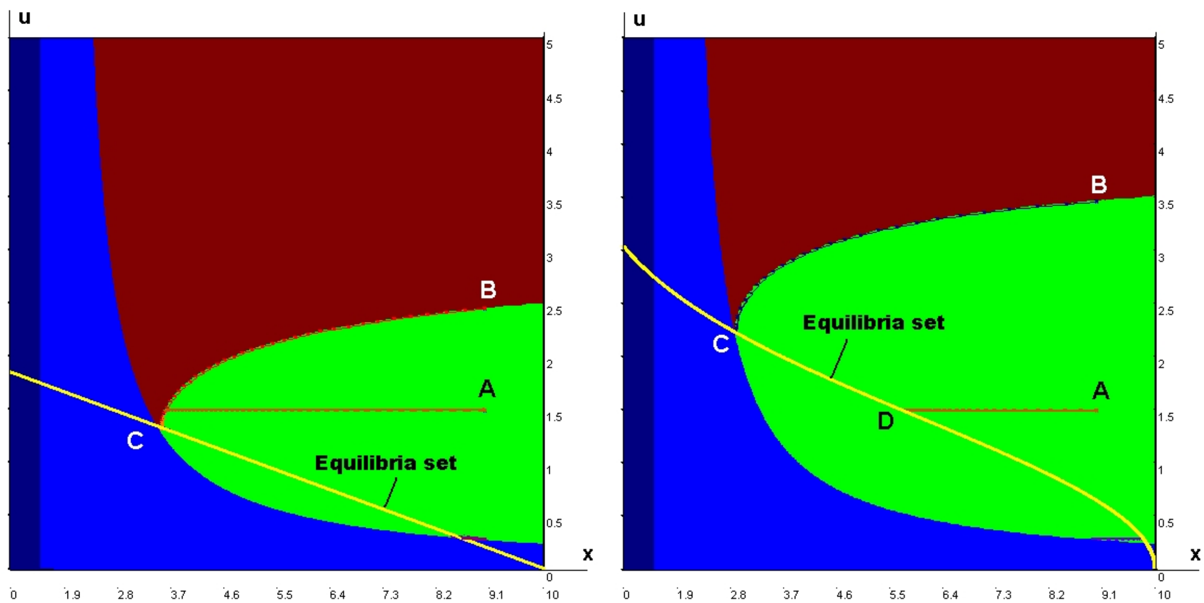


Figure 2.7: [Regulation Maps and heavy solutions under Verhulst-Schaeffer and Inert-Schaeffer Metasystems]
$x'(t) = rx(t)\left(1 - \frac{x(t)}{b}\right) - v(t)x(t)$ *and* $x'(t) = x(t)\left(r\sqrt{2\log\left(\frac{b}{x(t)}\right)} - v(t)\right)$ *respectively. The equilibrium lines are the graphs of* $r\left(1 - \frac{x}{b}\right)$ *and* $r\sqrt{2\log\left(\frac{b}{x}\right)}$*. Heavy evolutions stop when their trajectories hit the equilibrium line.*

Using the Malthusian (constant) feedbacks $\widetilde{u}_0(x) \equiv u$ for the growth of the renewable resource allows us to provide analytical formula of the inertia function for any decreasing exploitation function $\mathbf{v}(x)$ such as the Graham-Schaeffer one. Let us define by $\nu(u)$ the root of the equation $\mathbf{v}(x) = u$ and set $a(u) := \max(a, \gamma(u))$.

The inertia function is equal to:

$$\beta_u(x,v) := \begin{cases} \dfrac{(v-u)^2}{2\log\left(\frac{c}{a(u)}\right)} & \text{if} \quad v \geq u \text{ and } x \geq a(u) \\ 0 & \text{if} \quad \mathbf{v}(x) \leq v \leq u \text{ and } x \geq a(u) \end{cases}$$

The epigraph $\mathcal{E}p(\beta_u)$ of the inertia function is closed. However, its domain is neither closed nor open (and not even locally compact). The restriction of the inertia function to its domain is continuous.

Remark: — The inertia function is a solution to the Hamilton-Jacobi partial differential equation when $v \geq u$:

$$\frac{\partial \beta_u(x,v)}{\partial x}(u-v)x - \beta_u(x,v)\frac{\partial \beta_u(x,v)}{\partial v} = 0$$

Indeed, the partial derivatives of these two inertia functions are equal to

$$\frac{\partial \beta_u(x,v)}{\partial x} := -\frac{(v-u)^2}{2x\left(\log\left(\frac{x}{a(u)}\right)^2\right)} \;\&\; \frac{\partial \beta_u(x,v)}{\partial v} := \frac{v-u}{\log\left(\frac{x}{a(u)}\right)}$$

Observe that $\frac{\partial \beta_u(x,v)}{\partial v}$ is positive when $v > u$ and negative when $v < u$. ■

Proposition 2.2.2. *For system $x'(t) = (u - v(t))x(t)$, the* inert regulation map

$$(c,x) \rightsquigarrow R(c;x) := \{v \in \mathbb{R} \;\; \textit{such that}\; \beta_u(x,v) \leq c\}$$

associated with the inertia function is equal to

$$R(c,x) := \left[\mathbf{v}(x), u + \sqrt{2c\log\left(\frac{bx}{a(u)}\right)}\right] \text{ if } a(u) \leq x$$

The crisis map $(c,u) \rightsquigarrow \Xi(c;v) := \{x \in [a,b] \;\; \textit{such that}\; \beta_u(x,v) = c\}$ *is equal to*

$$\Xi((c,v)) = [a(u), \xi(c,v)] \text{ where } \xi(c,v) := a(u)e^{\frac{(v-u)^2}{2c}}$$

if $c > 0$ and to

$$\Xi(0,v) := \begin{cases} [a(u), +\infty[& \text{if } \mathbf{v}(x) \leq v \leq u \\ \emptyset & \text{if } v > u \end{cases}$$

if $c = 0$

The Viability Theorem provides the analytical formula of the metaregulation map $(x,v,c) \rightsquigarrow G(x,v,c)$ associating with any metastate (x,v,c) the set $G(x,v,c)$ of metaregulons govening the evolution of evolutions with finite inertia:

1. Case when $\beta_u(x,v) < c$. Then

$$G(x,v,c) \ := [-\beta_u(x,v), +\beta_u(x,v)]$$

2. Case when $\beta_u(x,v) = c$ and $v > u$. Then

$$G(x,v,c) := -\beta_u(x,v)$$

The minimal selection $g^0(x,v,c)$ is defined by $g^0(x,v,c) = 0$ if $\mathbf{v}(\mathbf{x}) \leq \mathbf{v} < \mathbf{u} + \sqrt{2c\log\left(\frac{bx}{a(u)}\right)}$ and by $g^0(x,v,c) = -\beta_u(x,v)$ whenever $v < u + \sqrt{2c\log\left(\frac{bx}{a(u)}\right)}$.

2.2.1 Inert Evolutions

An evolution $(x(\cdot), u(\cdot))$ is said to be inert *on a time interval $[t_0, t_1]$ if it is regulated by an affine open-loop controls of the form $v(t) := v + wt$, the velocities $v'(t) = vw$ of which are constant.*

The inertia function remains constant over an *inert evolution* as long as the evolution is viable: On an adequate interval, we obtain

$$\forall t \in [0, \overline{t}],\ \beta_u(\overline{x}(t), \overline{v}(t)) = \beta_u(x,v) = c$$

Let us consider the case when $v > u$.

The velocity governing the inert evolution is constant and equal to $\overline{v}'(t) = -\beta_u(x,v)$, so that

$$\overline{v}(t) = v\frac{(v-u)^2 t}{2\log\left(\frac{x}{a(u)}\right)}$$

and the evolution of the inert state by

$$\overline{x}(t) = xe^{-(v-u)t - \frac{(v-u)^2 t^2}{4\log\left(\frac{x}{a(u)}\right)}}$$

The state decreases until it reaches the lower bound $a(u)$ at time

$$\tau(x,v) = 2\frac{\log\left(\frac{x}{a(u)}\right)}{v-u}$$

and decreases until it reaches $a(u)$ in finite time.

This inert evolution is governed by the **inert feedback**

$$\widetilde{v}(x) := u + \sqrt{2 \log \left(\frac{x}{a(u)} \right)}$$

.

2.2.2 Heavy Evolutions

Heavy *evolutions* $x_c(\cdot)$ *are obtained when the absolute value* $|w(t)| := |v'(t)|$ *of the velocity* $w(t) := v'(t)$ *of the regulon is minimized at each instant. In particular, whenever the velocity of the regulon is equal to* 0*, the regulon is kept constant, and if not, it changes as slowly as possible.*

The "heaviest" evolutions are thus obtained by constant regulons. This is not always possible, because, by taking $v > u$ for instance, the solution $x(t) = xe^{-(v-u)t}$ is viable for $t \leq \frac{\log\left(\frac{x}{a(u)}\right)}{v-u}$. At that time, the regulon should be changed immediately (with infinite velocity) to any regulon $v \leq u$. This brutal and drastic measure — which is found in many natural systems — is translated in mathematics by impulse control.

In order to avoid such abrupt changes of regulons, *we add the requirement that the velocity of the regulons is bounded by a velocity bound* $c > \beta_u(x, v)$.

Starting from (x, v), the state $x_c(\cdot)$ of an heavy evolution evolves according

$$x_c(t) = xe^{-(v-u)t}$$

and reaches a at time $\frac{\log\left(\frac{x}{a(u)}\right)}{v-u}$.

The inertia function β_u provides the velocity of the regulons and increases over the heavy evolution according to

$$\forall t \in \left[0, \frac{\log\left(\frac{x}{a(u)}\right)}{v-u}\right], \ \beta_u(x_c(t), v) = \frac{(v-u)^2}{2\left(\log\left(\frac{x}{a(u))}\right) - (v-u)t\right)}$$

The derivatives of the inertia function over the inert evolutions are equal to

$$\frac{d\beta_u(x_c(t), v)}{dt} = \frac{(v-u)^3}{2\left(\log\left(\frac{x}{a(u)}\right) - (v-u)t\right)^2}$$

The inertia function reaches the given velocity limit $c > \beta_u(x,v)$ at

$$\begin{cases} \textbf{warning state} & \xi(c,u) = a(u)e^{\frac{(u-v)^2}{2c}} \\ \textbf{warning time} & \sigma_c(x,v) := \frac{\log\left(\frac{x}{a(u)}\right)}{v-u} - \frac{v-u}{2c} \end{cases}$$

Hence, once a velocity limit c is fixed, the heavy solution evolves with constant regulon u until the last instant $\sigma_c(x,v)$ when the state reaches $\xi_c(v)$ and the velocity of the regulon $\beta_u(\xi_c(v),v) = c$. *This is the last time when the regulon remains constant* and has to changed by taking

$$v_c(t) = v - c\left(t - \frac{\log\left(\frac{x}{a(u)}\right)}{v-u} + \frac{v-u}{2c}\right)$$

Then the evolution $(x_c(\cdot), v_c(\cdot))$ follows the inert solution starting at $(\xi_c(v),)$. It reaches equilibrium $(a(u), u)$ at time

$$t^\star := \frac{\log\left(\frac{x}{a(u)}\right)}{v-u} + \frac{v-u}{2c}$$

Taking $x(t) \equiv a(u)$ and $v(t) \equiv u$ when $t \geq t^\star$, the solution may remain at $a(u)$ forever.

For a given inertia bound $c > \beta_u(x,v)$, the *heavy evolution* $(x_c(\cdot), v_c(\cdot))$ is associated with the heavy feedback $\widetilde{v_c}$ defined by

$$\widetilde{v_c}(x) \ := \begin{cases} v & \text{if} \quad \xi(c,u) \leq x \\ u + \sqrt{2c\log\left(\frac{x}{a(u)}\right)} & \text{if} \quad a(u) \leq x \leq \ \xi(c,u) \end{cases}$$

If we add a constraint on the limitation of growth by introducing a carrying capacity b and by requiring that the interval $K := [a,b]$ is viable under the system $x'(t) = (u = v(t))x(t))$ where $v(t) \in V(x(t))$, we introduce the root $\nu(u)$ to the equation $\mathbf{v}(x) = u-$

Then the inertia function is equal to

$$\beta_u(x,v) \ := \begin{cases} \dfrac{(v-u)^2}{2\log\left(\frac{c}{a(u)}\right)} & \text{if} \quad v \geq u \text{ and } x \geq a(u) \\ -\dfrac{(v-u)^2}{2\log\left(\frac{b}{x}\right)} & \text{if} \quad \mathbf{v}(x) \leq v \leq u \text{ and } x \geq \nu(b) \end{cases}$$

2.2.3 Towards Dynamical Games

Actually, since we do not really know what are the dynamical equations governing the evolution of the resource, this suggest to leave open the choice of the growth rate of the resource and to regard it as a *tyche*.

Therefore, we just assume that the evolution of the resource is governed by the tychastic control system

$$x'(t) = (u(t) - v(t))x(t) \text{ where } u(t) \in U(x(t), \ v(t) \in V(x(t))$$

Let us denote by $\mathcal{P}(x, u, v)$ the set of evolutions governed by the above system. We also introduce a lower semicontinuous $(u, v) \mapsto \mathbf{n}(u, v)$ on the space $\mathcal{U} \times \mathcal{V}$, such as $\mathbf{n}(u, v) := \|u\| + \|v\|$.

Definition 2.2.3. *The guaranteed inertia function $\gamma(x, u, v)$ is defined by*

$$\gamma(x, u, v) := \inf_{\widetilde{v} \in \widetilde{\mathcal{V}}} \sup_{u((\cdot))} \sup_{t \geq 0} \mathbf{n}(u'(t), v'(t))$$

One can prove that the epigraph of the guaranteed inertia function is the guaranteed viability kernel of an auxiliary tychastic control system we shall define.

2.3 The Crisis Function

Figure 2.8: [Wei Ji = Danger-Opportunity]

In Chinese, the word crisis has an interesting meaning in terms of viability : The first ideogram, wei-xian, means "danger", the second one, "ji hui", means "opportunity". Thanks to our friend Shi Shuzhong for this piece of information. The Greek etymology "Krisis" means decision.

Consider a target $C \subset K$ contained in the environment K. Definition 1.11.4, p.45 of the perennial basin provides the subset on initial states of K from which starts at least one evolution reaching C in finite time and remaining C forever. In the same way than the hitting function $\varpi_{(K,C)}(x(\cdot))$ "quantifies" the capture basin by providing the minimum time needed for the evolution $x(\cdot)$ to reach the target C, the *crisis function* measures the time spent outside the subset C, i.e., the duration of *crisis* of not remaining in C.

In other words, if an evolution $x(\cdot)$ reaches the target in finite time at a point $x \in \partial K \cap \text{Viab}_{\mathcal{S}}(C)$, it will remain forever in C and, otherwise, if the evolution $x(\cdot)$ reaches C at a point

$x \in \partial K \backslash \mathrm{Viab}_{\mathcal{S}}(C)$, the evolution will leave C in finite time and enters a new era of crisis. This crisis may be endless if the evolution enters the complement of the capture basin $\mathrm{Capt}(K, C)$ of C. Otherwise, same scenario plays again.

Hence the complement $C \backslash \mathrm{Viab}_{\mathcal{S}}(C)$ can itself been partitioned in two subsets, one from which the evolutions will never return to the target (before leaving K), the other one from which at least one evolution returns and remains in the viability kernel of the target after a crisis lasting for a finite time of crisis. Luc Doyen and Patrick Saint-Pierre introduced and studied the concept of *crisis function to measure the time spent in* K *but outside* C *by evolutions* $x(\cdot) \in \mathcal{S}(x)$.

Definition 2.3.1. [**Crisis Function**] *The* crisis function $\upsilon_{(K,C)}(x) : X \mapsto \mathbf{R}_+ \cup \{+\infty\}$ *associates with* $x(\cdot)$ *its* crisis time *définie par*

$$\upsilon_{(K,C)}(x) := \inf_{x(\cdot) \in \mathcal{S}(x)} \mathrm{meas}\{t \geq 0 | x(t) \in K \backslash C\} = \inf_{x(\cdot) \in \mathcal{S}(x)} \int_0^{\infty} \chi_{K \backslash C}(x(\tau)) d\tau$$

where $\chi_{K \backslash C}$ *denotes the characteristic function of the complement of* $K \backslash C$.

The crisis function can be characterized in terms of the viability kernel of the auxiliary system:

$$\left\{ \begin{array}{ll} (i) & x'(t) = f(x(t), u(t)) \\ (ii) & y'(t) = -\chi_{K \backslash C}(x(t)) \\ & \text{where } u(t) \in U(x(t)) \end{array} \right. \tag{2.7}$$

subject to the constraint

$$\forall t \geq 0, \;\; (x(t), y(t)) \in K \times \mathbb{R}_+$$

Proposition 2.3.2. [**Viability Characterization of the Crisis Function**] *The crisis function is related to the viability kernel of* $K \times \mathbb{R}_+$ *under auxiliary system (2.7) by the following formula*

$$\upsilon_{(K,C)}(x) = \inf_{(x,y) \in \mathrm{Viab}_{(2.7)}(K \times \mathbb{R}_+)} y$$

Proof — Indeed, to say that (x, y) belongs to the viability kernel of $K \times \mathbb{R}_+$ under auxiliary system (2.7) amounts to saying that there exists an evolution $t \mapsto (x(t), y(t))$ governed by the auxiliary system such that, for all $t \geq 0$, $u(t) \in U(x(t))$. By definition of

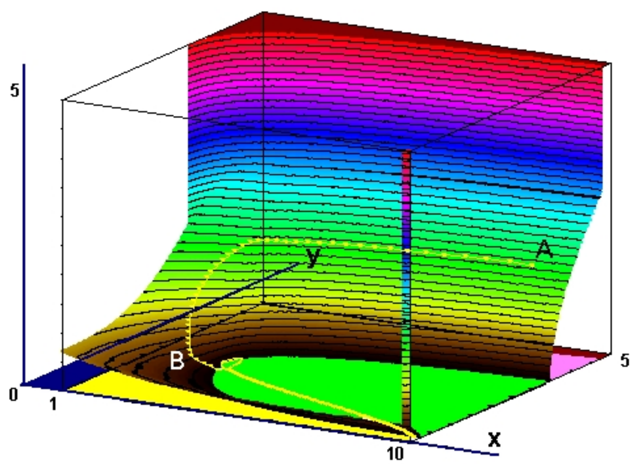
(2.7), we know that for all $t \geq 0$, this evolution also satisfies for all $t \geq 0$,

$$x(t) \in K \;\&\; y(t) = y - \int_0^t \chi_{K \setminus C}(x(\tau))d\tau \;\geq\; 0$$

Therefore

$$\sup_{t \geq 0} \left(\int_0^t \chi_{K \setminus C}(x(\tau)d\tau) \right) \leq y$$

and thus, $\upsilon_{(K,C)}(x) \leq \inf_{(x,y) \in \mathrm{Viab}_{(2.7)}(K \times \mathbb{R}_+)} y$. ■



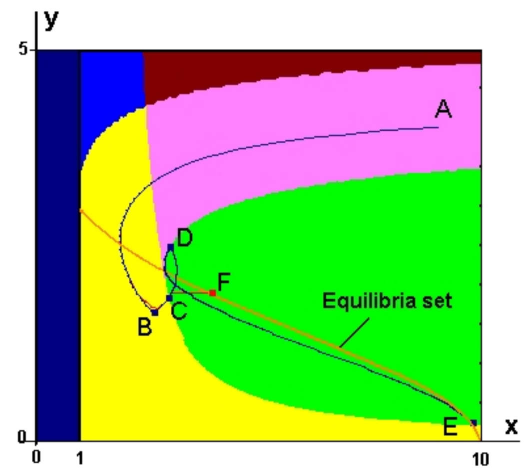


Figure 2.9: [Crisis Function]
Crisis Function under the Inert-Schaeffer Metasystem

The Inert-Schaeffer Metasystem $x'(t) = x(t)\left(\sqrt{\alpha}\sqrt{2\log(\frac{b}{x(t)})} - v(t)\right)$ modelling the evolution of renewable resources depleted by an economic activity $v(t)$. The metacontrols are the velocities $|v'(t)| \leq d$ of economic activity bounded by a constant d. The environment $\{(x, v) \in [a, b] \times [0, \overline{v}]$ | translates economic constraints. The figure of the left represents the graph of the crisis function, equal to zero on the viability kernel (in green), taking infinite values at states from which it is impossible to reach the environment. The figure of the right is its projection: the union of the black and deep blue areas is the complement of the domain of the crisis function. It is strictly positive and finite on states defined on the union of the purple and yellow areas. The yellow curve is the trajectory of the inert evolution. Figure 2.9 below deals with the crisis function under the

Inert-Schaeffer metasystem $x'(t) = x(t)\left(\sqrt{\alpha}\sqrt{2\log\left(\frac{b}{x(t)}\right)} - v(t)\right)$.

2.4 Global Climate Change

We illustrate this problem with the simplest example. Many difficulties of collective decision-making models about climatic risks are due to the interactions between physical and economical requirements. The precaution principle and economic efficiency are often contradictory.

In order to stylize the problems for providing a two-dimensional illustration, they isolated two variables:

1. the concentration $x(t) \in [0, b]$ of green-house gases — say, CO_2 — regarded as a state variable, bounded by a given constant b
2. the short-term pollution rate (generated when using a given technology and a level of production) $u(t) \in \mathbb{R}_+$, regarded as regulons.

The *ecological constraints* being represented by the interval $[0, b]$, the *economic constraints* amounts to bound or minimize a transition cost measured in this example by the absolute value of the velocity of the pollution rate. How this cost is a new constraint for a macro-economic model is another question which is not treated in the study of this simple example.

We assume that the evolution of the concentration of green-house gases is governed by

$$x'(t) = u(t) - ax(t)$$

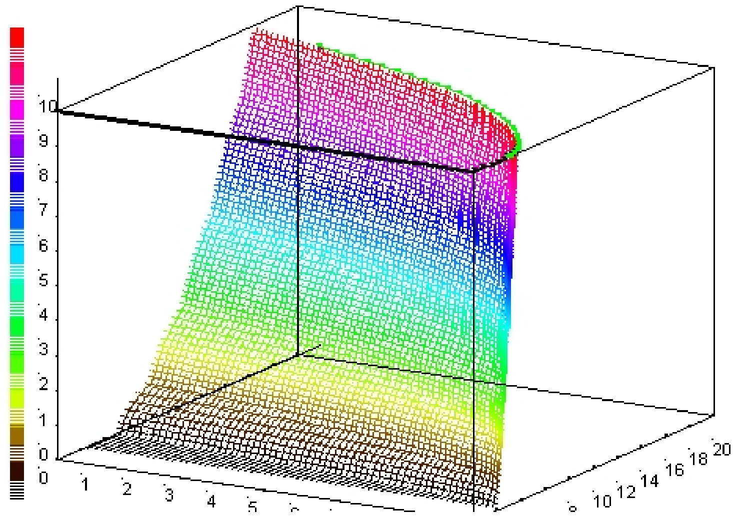
That means that the variation of the concentration of the green-house gases depends upon a natural slow absorption phenomenon by the oceans ($-ax(t)$ with a "small") and is proportional to the short-term pollution.

We denote by $\mathcal{P}(x, u)$ the set of evolutions governed by $x'(t) = u(t) - ax(t)$ viable in $[0, b] \times \mathbb{R}_+$. Hence, the inertia function is defined by

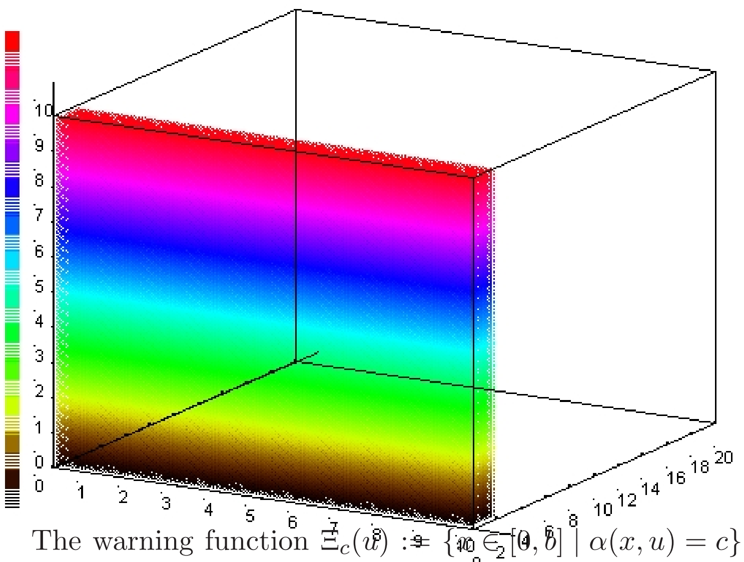
$$\alpha(x, u) := \inf_{(x(\cdot),u(\cdot))\in\mathcal{P}(x,u)} \sup_{t\geq 0} |u'(t)|$$

One can compute this inertia function with the viability kernel algorithm (see Figure 2.10) and also compute it explicitly: $\alpha(x, u)$ is the implicit solution to the equation

$$a(x - au) = \alpha(x, u)\left(1 - e^{\frac{a}{\alpha(x,u)}(u-ab)}\right)$$



Inertia Function $\alpha(x,u) := \inf_{u(.)} \sup_{t\geq 0} |u'(t)|$ of system $x'(t) = u(t) - ax(t)$ on the interval $[0,1]$. One can regard the control u as the pollution rate, the state x as the mass of green-house gazes and the inertia function as the minimaximal intertemporal transition cost of changing of pollution rate. Source: Patrick Saint-Pierre



The inertia set is the level set of the inertia function $\alpha(x,u) := 0$ of system $x'(t) = u(t) - ax(t)$ on the interval $[0,1]$. It is here the product $[0,1] \times [0,ab]$. In this set, the state can be governed by passive evolutions. Source: Patrick Saint-Pierre.

Figure 2.11: [Inertia Set]

The warning function $\Xi_c(u) := \{x \in [0,b] \mid \alpha(x,u) = c\}$ is then equal a single-valued function $\mathbb{R}_+ \mapsto \mathbb{R}$ defined by

$$\Xi_c(u) = \frac{c}{a^2}\left(1 - e^{\frac{a}{c}(u-ab)}\right) + \frac{u}{a}$$

It satisfies $\Xi_c(ab) = b$ and $\xi_c(0) = \frac{c}{a^2}\left(1 - e^{\frac{-a^2 b}{c}}\right)$, which provides the smallest concentration of green-house gas that we can obtain by choosing the most drastic reduction strategy using $u'(t) = -c$.

We observe that $\alpha(x,u) = 0$ if and only if $(x,u) \in [0,b] \times [0,ab]$. In this case, passive evolutions $x_u(\cdot)$ are viable in $[0,b]$ and converge to the equilibrium $\frac{u}{a}$, which is stable whenever $u \in [0,ab]$.

The level sets are given by the formula

$$\{(x,u) \in [0,b] \times \mathbb{R}_+ \mid \alpha(x,u) \le c\} = \{(x,u) \in [0,b] \times \mathbb{R}_+ \mid x \le \Xi_c(u)\}$$

The trajectories of inert evolutions $(x(\cdot), u(\cdot))$ satisfying $\alpha(x(t), u(t)) = c$ satisfy the equation $x(t) = \Xi_c(u - ct)$.

The *heavy evolution* consists in keeping the same pollution as long as the mass of green-house gas is smaller than $\Xi_c(u)$. At this level, the technology has to be drastically changed with the velocity equal to $-c$, while the concentration of green-house gas increases until it reaches the level b, which is an equilibrium where the heavy evolution stops.

As a second example, we can add a macro-economic interaction, stating that the emissions of pollutants depend upon the economic activity $z(t)$. At his very elementary level of illustration of the phenomenon, we assume that we have access to the velocity of the pollution rate through a bound of the form $|y'(t)| \le z(t)$ set by economic activity $z(t)$. This ignorance is taken into account by the "meta-inertia function" that we now define.

We denote by $\mathcal{P}(x,y,z)$ the subset of "meta-evolutions" $(x(\cdot), y(\cdot), z(\cdot))$ starting at $x(0) = x$,$y(0) = y$ and $z(0) = z$, viable in $[0,b] \times \mathbb{R}_+ \times \mathbb{R}$ where $(x(\cdot), y(\cdot))$ is governed by

$$\begin{cases} (i) & x'(t) = y(t) - axt) \\ (ii) & |y'(t)| \le z(t) \end{cases}$$

The "meta-inertia function" $(x,y,z) \mapsto \beta(x,y,z)$ associated with this meta-system is defined by

$$\beta(x,y,z) := \inf_{(x(\cdot),y(\cdot),z(\cdot)) \in \mathcal{P}(x,y,z)} \sup_{t \ge 0} |z'(t)|$$

Since the graph of this function is 4-dimensional, we shall represent it by its 3-dimensional level-sets

$$\{(x,y,z) \in [0,b] \times \mathbb{R}_+ \times \mathbb{R} \text{ such that } \beta(x,y,z) \le c\}$$

We observe that the inertia function α is related to the meta-inertia function β by the relations

$$\text{Graph}(\alpha) = \{(x,y,z) \in [0,b] \times \mathbb{R}_+ \times \mathbb{R} \text{ such that } \beta(x,y,z) \;\le\; 0\}$$

Indeed, we remark that the inertia $z := \alpha(x,y)$ of an evolutions $(x(\cdot), y(\cdot)) \in \mathcal{P}(x,y)$ is finite if and only if, setting $z(t) \equiv z$, the meta-evolution $(x(\cdot), y(\cdot), z(\cdot)) \in \mathcal{P}(x,y,z)$ satisfies

$$\beta(x,y,z) := \inf_{(x(\cdot),y(\cdot),z(\cdot)) \in \mathcal{P}(x,y,z)} \sup_{t \ge 0} |z'(t)| = 0 \;\blacksquare$$

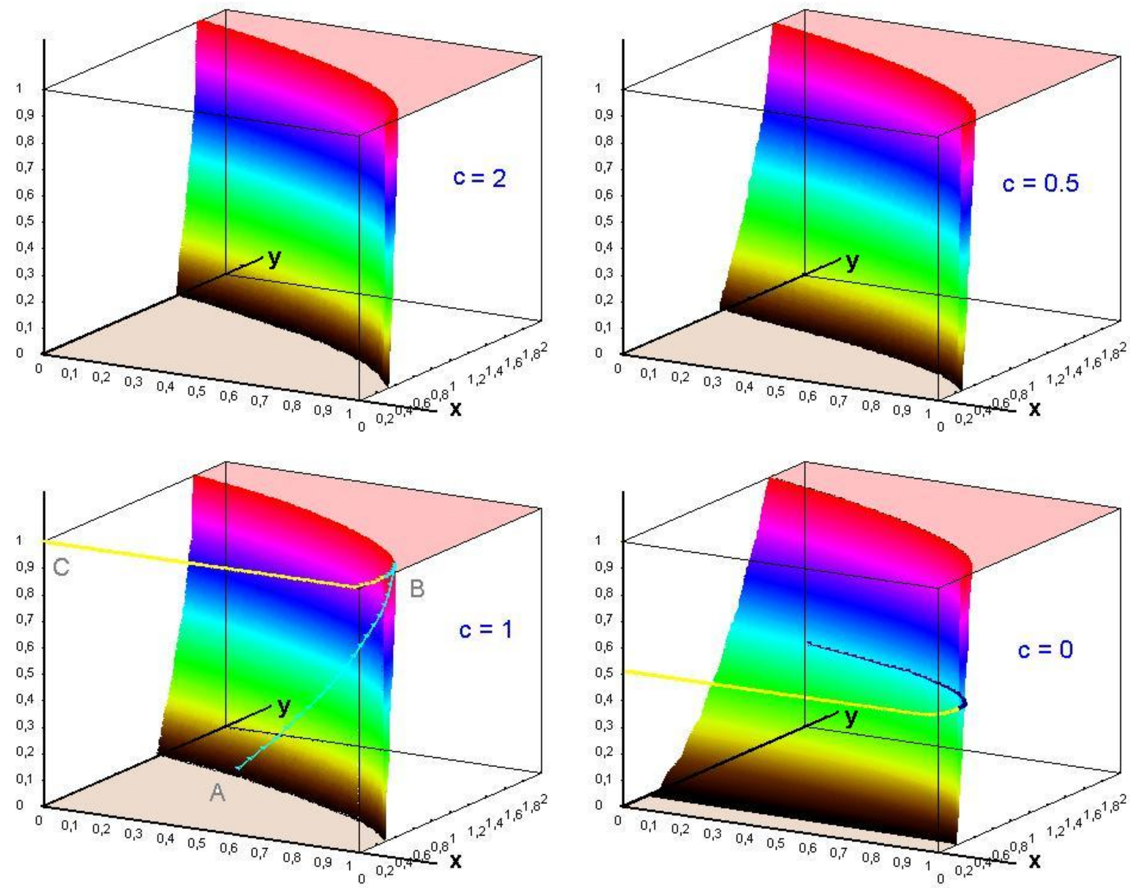


Figure 2.4.1. Level-sets of the Meta-Inertia Function. $\beta(x,y,z) := inf_{(x(\cdot),y(\cdot),z(\cdot))\in\mathcal{P}(x,y,z)} \sup_{t\geq 0} |z'(t)|$ *when the evolution* $x(\cdot)$ *is governed by the system* $x'(t) = y(t) - 0.2x(t)$ *and* $|y'(t)| \leq z(t)$ *defined on* $[0,1]\times[0,25]\times[-1,+1]$ *for several values of* $c = 0,\ 0.5,\ 1\ \&\ 2$. *We assume only that the derivative* $y'(\cdot)$ *of the pollution rate* $y(\cdot)$ *is bounded by a measure of the economic activity. The meta-inertia function provides the minimaximal intertemporal economic transition cost of changing of pollution rate. For* $c = 0$*, we find the graph of the inertia function* α*, equal to the level set* $\{(x,y,z) \in [0,b]\times\mathbb{R}_+\times\mathbb{R}$ *such that* $\beta(x,y,z) \leq 0\}$*. The trajectory of the evolution represented for* $c = 2$ *is an heavy evolution (for* $z := \alpha(x,y)$*) and the evolution represented for* $c = 0$ *is an inert one. For* $c = 1$*, the trajectory of the evolution starting from* B *is an heavy evolution of the level set* $\{(x,y,z) \in [0,b]\times\mathbb{R}_+\times\mathbb{R}$ *such that* $\beta(x,y,z) \leq 1\}$ *arriving at equilibrium* A*.*

We consider now the backward dynamical system

$$\text{for almost all } t \geq 0,\ x'(t) = \alpha x(t) - u(t), \ \text{ where } \ u(t) \geq 0$$

subjected to the constraints

$$\forall t \geq 0,\ x(t) \in [0, b]$$

We may regard $K := [0, b]$ as the subset of a scarce commodity x regarded as the state and the above dynamics as the dynamics of a greedy consumer where her consumption is slowed down by a price regarded as a regulon.

We see at once that the viable equilibria of the system range over the *equilibrium line* $u = \alpha x$ and that they are *unstable*. The viability niche of the price $\bar{u}$ is reduced to its associated equilibria $\dfrac{\bar{u}}{\alpha}$.

The regulation map is given by the formula

$$R_K(0) = \{0\},\ R_K(x) = \mathbf{R}_+ \ \text{ when } \ \ x \in]0, b[\ \&\ R_K(b) = [\alpha b, +\infty[$$

When the inflation is bounded, the evolution of the consumption-price pair is governed by the system of differential inclusions

$$\left\{ \begin{array}{ll} (i) & \text{for almost all } t \geq 0,\ x'(t) = \alpha x(t) - u(t) \\ (ii) & \text{and } \ -c \leq u'(t) \leq c \end{array} \right. \tag{2.8}$$

which are viable in $[0, b] \times \mathbf{R}_+$.

We introduce the functions $\rho_c^\sharp$ and $\rho_c^\flat$ defined on $[0, \infty[$ by

$$\left\{ \begin{array}{ll} (i) & \rho_c^\flat(u) := \frac{c}{\alpha^2}(e^{-\alpha u/c} - 1 + \frac{\alpha}{c}u) \approx \frac{u^2}{2c} \\ \\ (ii) & \rho_c^\sharp(u) := -ce^{\alpha(u-\alpha b)/c}/\alpha^2 + u/a + c/\alpha^2 \end{array} \right.$$

and the functions $r_c^\sharp$ and $r_c^\flat$ defined on $[0, b]$ by

$$\left\{ \begin{array}{ll} (i) & r_c^\flat(x) = u \ \text{ if and only if } \ x = \rho_c^\flat(u) \\ \\ (ii) & r_c^\sharp(x) = 0 \ \text{ if } \ x \in [0, \rho_c^\sharp(0)]\ (\rho_c^\sharp(0) = \frac{c}{\alpha^2}(1 - e^{-\alpha^2 b/c})) \\ \\ (iii) & r_c^\sharp(x) = u \ \text{ if and only if } \ x = \rho_c^\sharp(u) \ \text{ when } \ x \in [\rho_c^\sharp(0), b] \end{array} \right.$$

We introduce now the set-valued map R^c defined by

$$\forall x \in [0, b],\ R^c(x) = [r_c^\sharp(x), r_c^\flat(x)] \tag{2.9}$$

There exist solutions to (2.8) if and only if the initial state satisfies $u_0 \in R^c(x_0)$*. In this case, prices and commodities are related by the new pricing law*:

$$\forall t \geq 0,\ u(t) \in R^c(x(t))$$

Starting from a consumption-price pair within the viability kernel and below the equilibrium line, the heavy solution starts by evolving at zero inflation until the consumption-price pair hits the inflationist curve going through $B := (b, \alpha b)$. Then the price must increase with the highest velocity until the consumption-price pair reaches the equilibrium B. ■

Chapter 3

Other Kernels and Basins

3.1 Tychastic Systems

The questions involved in the concepts of viability kernels and capture basins ask only of the existence of an evolution satisfying the viability or the viability/capturability issue. In the case of parameterized systems, this lead to the interpretation of the parameter as a control or a regulon. When the parameters are regarded as tyches, disturbances, perturbations, etc., the questions are dual : they require that all evolutions satisfy the viability or the viability/capturability issue.

Figure 3.1: [Tyche]

Uncertainty without statistical regularity can be translated mathematically by parameters on which actors, agents, decision makers, etc. have no controls. These parameters are often perturbations, disturbances (as in "robust control" or "differential games against nature") or more generally, tyches *(meaning "chance" in classical Greek, from the Goddess Tyche) ranging over a state-dependent* tychastic map*. They could have be called "random variables" if this vocabulary were not already confiscated by probabilists. This is why we borrow the term of* tychastic evolution *to* Charles Peirce *who introduced it in* 1893 *under the title evolutionary love.*

A *tychastic system* is a parameterized differential equation

$$x'(t) = f(x(t), v(t)) \text{ where } v(t) \in V(x(t)) \tag{3.1}$$

where the parameters $v(t)$ are regarded as tyches. Such a system is called a *tychastic system* and the set-valued map V the *tychastic map*.

Even though the parameterized system is written in the same way than the regulated system (3.1), the roles played by controls u and tyches v and the questions are completely different, actually, "dual" in some sense. Instead of requiring the existence of *at least* one control or regulon $u(\cdot)$ such that the associated evolution satisfies a required property (viability, capturability, optimality, etc.), we ask that *for all* tyches $v(t)$, the associated evolutions governed by the tychastic system do satisfy this given property.

3.2 Invariance Kernel under a Tychastic System

When the parameterized system is regarded as a tychastic system, it is natural to consider the core of a set of evolutions under a tychastic system:

Definition 3.2.1. [**Core under an Evolutionary System**] *Let $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0,\infty;X)$ denote an evolutionary system and $\mathcal{H} \subset \mathcal{C}(0,\infty;X)$ a subset of evolutions sharing a given property. The set*

$$\mathcal{S}^{\ominus 1}(\mathcal{H}) := \{x \in X \mid \mathcal{S}(x) \ \subset \ \mathcal{H}\} \tag{3.2}$$

*of initial states $x \in X$ from which **all evolutions** $x(\cdot) \in \mathcal{S}(x)$ satisfy the property $\mathcal{H}$ is called the* core *of $\mathcal{H}$ under $\mathcal{S}$.*

Taking $\mathcal{H} := \mathcal{V}(K,C)$, we obtain the *invariance kernel*(or *tychastic viability kernel*) $\mathrm{Inv}_{\mathcal{S}}(K,C) := \mathrm{Tych}_{\mathcal{S}}(\mathcal{V}(K,C))$ of K outside C:

Definition 3.2.2. [**Invariance Kernel and Absorption Basin**] *Let $K \subset X$ be a environment and $C \subset K$ be a target.*

1. *The subset $\mathrm{Inv}_{\mathcal{S}}(K,C)$ of initial states $x_0 \in K$ such that **all evolutions** $x(\cdot) \in \mathcal{S}(x_0)$ starting at x_0 are viable in K for all $t \geq 0$ or viable in K until they reach C in finite time is called the* invariance kernel of K with target C *under $\mathcal{S}$.*

   When the target $C = \emptyset$ is the empty set, we say that $\mathrm{Inv}_{\mathcal{S}}(K) := \mathrm{Inv}_{\mathcal{S}}(K,\emptyset)$ is the invariance kernel *of K.*

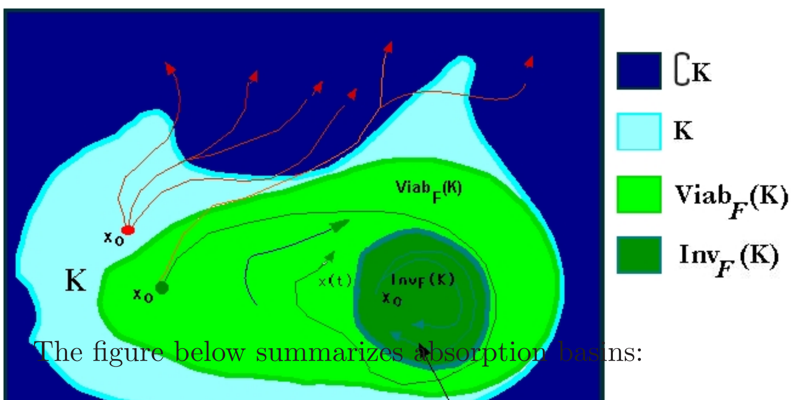
2. *The subset $\mathrm{Abs}_{\mathcal{S}}(K,C)$ of initial states $x_0 \in K$ such that **all evolutions** $x(\cdot) \in \mathcal{S}(x_0)$ starting at x_0 are viable in K until they reach C in finite time is called the* absorption basin of K with target C *under $\mathcal{S}$.*

   When $K = X$ is the whole space, we say that $\mathrm{Abs}_{\mathcal{S}}(X,C)$ is the absorption basin *of*

C.

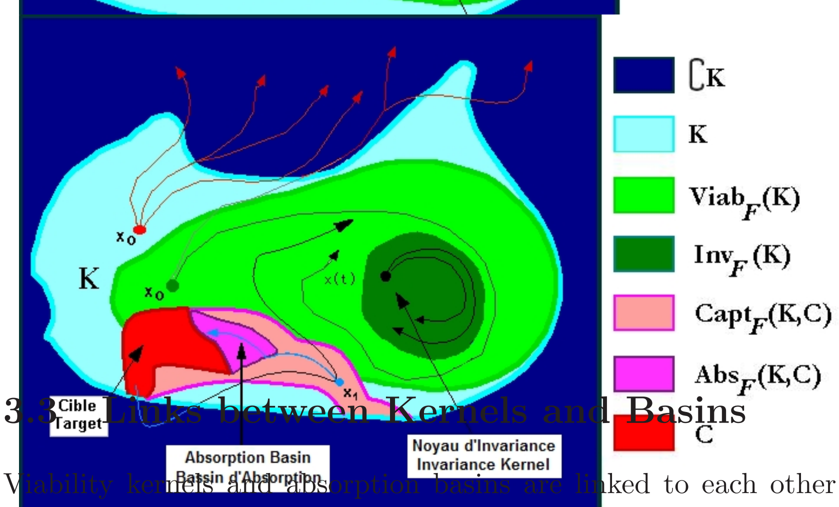
We say that

1. *a subset* K *is* invariant *under* $\mathcal{S}$ *if* $K = \text{Inv}(K)$,
2. K *is* invariant outside a target $C \subset K$ *under the evolutionary system* $\mathcal{S}$ *if* $K = \text{Inv}(K, C)$,
3. C *is* separated *in* K *if* $C = \text{Inv}(K, C)$.



*A state x_1 belongs to the invariance kernel of the environment K under an evolutionary system if **all** the evolutions starting from it are viable in K forever. Starting from a state $x_2 \in K$ outside the invariance kernel, **at least** one evolution leaves the environment in finite time.*

The figure below summarizes absorption basins:



***All** evolutions starting from a state x_1 in the absorption basin of the target C invariant in the environment K are viable in K until they reach C in finite time. **At least** one evolution starting from $x_2 \in K$ outside the absorption basin remain outside the target C until it leaves K.*

Figure 3.3: [**Schema of an Absorption Basin.**]

3.3 Links between Kernels and Basins

Viability kernels and absorption basins are linked to each other by complementarity, as well as invariance kernels and capture basins:

Definition 3.3.1. [**Complement of a Subset**] *The complement of the subset $C \subset K$ in K is the set $K\backslash C$ of elements $x \in K$ not belonging to C. When $K := X$ is the whole space, we set $\complement\, C := X\backslash C$. Observe that*

$$K\backslash C = \complement C\backslash\complement K \text{ and } \complement(K \setminus C) = C \cup \complement K$$

The following useful consequences relating the kernels and basins follow readily from the definitions:

Lemma 3.3.2. [**Complements of Kernels and Basins**] *Kernels and Basins are exchanged by complementarity:*

$$\left\{ \begin{array}{ll} (i) & \complement\mathrm{Viab}_{\mathcal{S}}(K, C) = \mathrm{Abs}_{\mathcal{S}}(\complement C, \complement K) \\ (ii) & \complement\mathrm{Capt}_{\mathcal{S}}(K, C) = \mathrm{Inv}_{\mathcal{S}}(\complement C, \complement K) \end{array} \right. \tag{3.3}$$

3.4 Tychastic and Stochastic Invariance

The reason to use this terminology is to underline the comparison with stochastic differential equations. Concepts of invariance kernels and absorption basins can also be defined for usual stochastic differential equations. : Stochastic invariance kernels and absorption basins are particular cases of invariance kernels and absorption basins under systems of the form (1.10).

To be precise, let us consider *random events* $\omega \in \Omega$, where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, instead of tyches $v(\cdot) \in V(x(\cdot))$.

Denote by $\mathbb{X}^x_\omega(t) := \mathbb{X}(x, \omega)(t)$ the solution starting at x to the stochastic differential equation

$$dx = \gamma(x)dt + \sigma(x)dW(t) \tag{3.4}$$

where $W(t)$ ranges over a finite dimensional vector space $Y \subset X$, the drift $\gamma : X \mapsto X$ and the diffusion $\sigma : X \mapsto \mathcal{L}(Y, X)$ are smooth and bounded maps. In other words, it defines evolutions $t \mapsto \mathbb{X}(x, \omega)(t) := \mathbb{X}^x_\omega(t) \in X$ starting at x at time 0 and parameterized by random events $\omega \in \Omega$ satisfying technical requirements (measurability, filtration, etc.) that are not relevant to involve at this stage of the exposition. The initial state x being fixed, the random variable $\omega \mapsto \mathbb{X}(x, \omega) := \mathbb{X}^x_\omega(\cdot) \in \mathcal{C}(0, \infty; X)$ is called a stochastic process.

When a subset $\mathcal{H} \subset \mathcal{C}(0,\infty;X)$ of prescribed evolutions is a closed subset, we denote by $\mathbb{P}_{\mathbb{X}(x,\cdot)}$ the law of the random variable $\mathbb{X}(x,\cdot)$ defined by

$$\mathbb{P}_{\mathbb{X}(x,\cdot)}(\mathcal{H}) \ := \ \mathbb{P}(\{\omega \mid \mathbb{X}(x,\omega) \in \mathcal{H}\}) \tag{3.5}$$

It is natural to introduce the stochastic core of $\mathcal{H}$ under the stochastic system: It is the subset of initial states x from which starts a stochastic process $\omega \mapsto \mathbb{X}(x,\omega)$ such that for almost all $\omega \in \Omega$, $\mathbb{X}(x,\omega) \in \mathcal{H}$:

$$\mathrm{Stoc}_{\mathbb{X}}(\mathcal{H}) \ := \ \{x \in X \mid \text{for almost all } \omega \in \Omega, \ \ \mathbb{X}(x,\omega) := \mathbb{X}^x_\omega(\cdot) \ \in \ \mathcal{H}\} \tag{3.6}$$

On the other hand, let us associate with drift and the diffusion the Stratonovitch drift $\widehat{\gamma}$ defined by $\widehat{\gamma}(x) := \gamma(x) - \frac{1}{2}\sigma'(x)\sigma(x)$.

We associate with this stochastic differential equation the specific tychastic system

$$\left\{ \begin{array}{lll} (i) & x'(t) \ = \ \widehat{\gamma}(x(t)) + \sigma(x(t))v(t) \\ (ii) & v(t) \ \in \ Y \end{array} \right. \tag{3.7}$$

where the tychastic map is constant and equal to Y.

Denoting by $\mathcal{S}$ the evolutionary system associated with tychastic system (3.7), we can associate with $\mathcal{H}$ its tychastic core defined by the subset

$$\mathcal{S}^{\ominus 1}(\mathcal{H}) \ := \ \{x \in X \mid \mathcal{S}(x) \ \subset \ \mathcal{H}\} \tag{3.8}$$

of initial states from which all evolutions governed by (3.7) belong to $\mathcal{H}$.

The definitions of the tychastic and stochastic cores of subsets of evolution properties are similar in spirit.

But there is a deeper similarity that we mention briefly: The Strook-Varadhan Support Theorem implies that whenever $\mathcal{H}$ is closed, the stochastic core of $\mathcal{H}$ under the stochastic system $\mathbb{X}$ and its tychastic core under the associated tychastic system $\mathcal{S}$ coincide:

$$\mathrm{Stoc}_{\mathbb{X}}(\mathcal{H}) \ = \ \mathcal{S}^{\ominus 1}(\mathcal{H})$$

Taking for subset $\mathcal{H}$ the closed subset of evolutions viable in a closed environment K forever or until it captures a closed target $C \subset K$, we deduce that *the stochastic viability kernel with target under stochastic differential equation (3.4) is equal to the invariance kernel with target under the associated tychastic system (3.7).*

Furthermore, *the tychastic system associated with a stochastic one by the Strook-Varadhan Support Theorem is very particular: there is no bound on the tyches, whereas general tychastic* *systems allow the tyches to range over subsets $V(x)$ depending upon the state x, describing so to speak a state-dependent volatility.* This state-dependent uncertainty, unfortunately absent in the mathematical representation of uncertainty in the framework of stochastic processes, is of utmost importance for describing uncertainty in problems dealing with living beings.

Remark: The Strook-Varadhan Support Theorem — When a subset $\mathcal{H} \subset \mathcal{C}(0,\infty;X)$ of prescribed evolutions is a closed subset, we denote by $\mathbb{P}_{\mathbb{X}(x,\cdot)}$ the law of the random variable $\mathbb{X}(x,\cdot)$ defined by

$$\mathbb{P}_{\mathbb{X}(x,\cdot)}(\mathcal{H}) \ := \ \mathbb{P}(\{\omega \mid \mathbb{X}(x,\omega) \in \mathcal{H}\}) \tag{3.9}$$

Therefore, we can reformulate the definition of the stochastic core of a set $\mathcal{H}$ of evolutions in the form

$$\mathrm{Stoc}_{\mathbb{X}}(\mathcal{H}) \ = \ \{x \in X \mid \mathbb{P}_{\mathbb{X}(x,\cdot)}(\mathcal{H}) = 1\} \tag{3.10}$$

In other words, the stochastic core of $\mathcal{H}$ is the set of initial states x such that the subset $\mathcal{H}$ has probability one under the law of the stochastic process $\omega \mapsto \mathbb{X}(x,\omega) \in \mathcal{C}(0,\infty;X)$ (if $\mathcal{H}$ is closed, $\mathcal{H}$ is called the support of the law $\mathbb{P}_{\mathbb{X}(x,\cdot)}$). The Strook-Varadhan Support Theorem states that under regularity assumptions, this support is the core of $\mathcal{H}$ under tychastic system (3.7).

■

3.5 Viability and Invariance Kernels of Tubes

Let us consider time-dependent system

$$\left\{ \begin{array}{ll} (i) & x'(t) = f(t,x(t),u(t)) \\ (ii) & u(t) \in U(t,x(t)) \end{array} \right. \tag{3.11}$$

Definition 3.5.1. [**Viability Kernel of a Tube**] *Let us consider a time-dependent environment $K(t)$, regarded as a tube $K(\cdot)$. The T-viability kernel* $\mathrm{Viab}_{(3.11)}(K)(T)$ *of the tube $K(\cdot)$ under system 3.11, p.105 is the set of initial states $x \in K(0)$ from which starts at least one evolution governed by (3.11) viable in the tube on the interval $[0,T]$ in the sense that*

$$\forall\, t \in [0,T], \ \ x(t) \in K(t)$$

The T-invariance kernel $\mathrm{Inv}_{(3.11)}(K)(T)$ *of the tube* $K(\cdot)$ *under system 3.11, p.105 is the set of initial states* $x \in K(0)$ *such that all solutions starting at* x *and governed by (3.11) are viable in the tube on the interval* $[0,T]$.

We associate with any T the

- tube $\widehat{K}_T(t) := K(T-t)$ and its graph $\mathrm{Graph}(\widehat{K}_T) \subset \mathbb{R} \times X$
- the auxiliary dynamic system

$$\left\{ \begin{array}{ll} (i) & \tau'(t) \;=\; -1 \\ (ii) & x'(t) = f(T-\tau(t), x(t), u(t)) \\ (iii) & u(t) \in U(T-\tau(t), x(t)) \end{array} \right. \tag{3.12}$$

Proposition 3.5.2. [**The Graph of the Viability and Invariance Kernels of Tubes**]

1. *The graph of the T- viability kernel of the tube* $\mathrm{Viab}_{(3.11)}(K)(\cdot)$ *is the viable-capture basin of* $\{0\} \times \widehat{K}_T(0)$ *viable in* $\mathrm{Graph}(\widehat{K}_T)$ *under the evolutionary system (3.12):*

$$\mathrm{Graph}(\mathrm{Viab}_{(3.11)}(K)(\cdot)) = \mathrm{Capt}_{(3.12)}(\mathrm{Graph}(\widehat{K}_T), \{0\} \times K(0))$$

2. *The graph of the T- invariance kernel of the tube* $\mathrm{Inv}_{(3.11)}(K)(\cdot)$ *is the viable-absorption basin of* $\{0\} \times \widehat{K}_T(0)$ *viable in* $\mathrm{Graph}(\widehat{K}_T)$ *under the evolutionary system (3.12):*

$$\mathrm{Graph}(\mathrm{Inv}_{(3.11)}(K)(\cdot)) = \mathrm{Abs}_{(3.12)}(\mathrm{Graph}(\widehat{K}_T), \{0\} \times K(0))$$

3.6 Inverse of Viability and Invariance Kernels

3.6.1 Vector parameters

Let us consider control systems

$$x'(t) \;\in\; F(\lambda, x(t)) \tag{3.13}$$

environments $K(\lambda)$ and targets $C(\lambda)$ depending upon a parameter $\lambda \in \Lambda$ ranging over a finite dimensional vector space Λ.

The problem is to *invert* the set-valued maps

$$\mathbb{V} : \lambda \rightsquigarrow \text{Viab}_{F(\lambda,\cdot)}(K(\lambda), C(\lambda)) \text{ and } \mathbb{I} : \lambda \rightsquigarrow \text{Inv}_{F(\lambda,\cdot)}(K(\lambda), C(\lambda))$$

For that purpose, we shall characterize the graphs of these maps:

Theorem 3.6.1. Inverse Viability Kernel. *The graph of the map* $\mathbb{V} : \lambda \rightsquigarrow \text{Viab}_{F(\lambda,\cdot)}(K(\lambda), C(\lambda))$ *is equal to the viability kernel*

$$\text{Graph}(\mathbb{V}) = \text{Viab}_{(3.14)}(\mathcal{K}, \mathcal{C})$$

of the graph $\mathcal{K} := \text{Graph}(\lambda \rightsquigarrow K(\lambda))$ *with target* $\mathcal{C} := \text{Graph}(\lambda \rightsquigarrow C(\lambda))$ *under the auxiliary system*

$$\left\{ \begin{array}{lll} (i) & \lambda'(t) & = 0 \\ (ii) & x'(t) & \in F(\lambda, x(t)) \end{array} \right. \tag{3.14}$$

In the same way, the graph of the map $\mathbb{I} : \lambda \rightsquigarrow \text{Inv}_{F(\lambda,\cdot)}(K(\lambda), C(\lambda))$ *is equal to the invariance kernel*

$$\text{Graph}(\mathbb{I}) = \text{Inv}_{(3.14)}(\mathcal{K}, \mathcal{C})$$

of the graph $\mathcal{K} := \text{Graph}(\lambda \rightsquigarrow K(\lambda))$ *with target* $\mathcal{C} := \text{Graph}(\lambda \rightsquigarrow C(\lambda))$ *under the auxiliary system (3.14).*
Consequently,the inverses $\mathbb{V}^{-1}$ *and* $\mathbb{I}^{-1}$ *of the set-valued maps* $\mathbb{V}$ *and* I *associate with any* $x \in X$ *the subsets of parameters* $\lambda \in \Lambda$ *such that the pairs* (λ, x) *belong to the viability and invariance kernels of the graph* $\mathcal{K} := \text{Graph}(\lambda \rightsquigarrow K(\lambda))$ *with target* $\mathcal{C} := \text{Graph}(\lambda \rightsquigarrow C(\lambda))$ *under the auxiliary system (3.14) respectively.*

Under adequate Marchaud and Lipschitz assumptions, the inverse of the set-valued map $\mathbb{V}^{-1}$ is the unique solution to the system of partial differential inclusions

$$\left\{ \begin{array}{ll} (i) & \forall\, \lambda \in \mathbb{V}^{-1}(x) \setminus C^{-1}(x), \;\; 0 \in \bigcup_{v \in F(\lambda,x)} D\mathbb{V}^{-1}(v) \\ (ii) & \forall\, \lambda \in \mathbb{V}^{-1}(x), \;\; 0 \in \bigcap_{v \in -F(\lambda,x) \cap DK(\lambda,x)(0)} D\mathbb{V}^{-1}(v) \end{array} \right.$$

3.6.2 Scalar parameters

Proposition 3.6.2. Viability Functional. *Let us consider the case when $\Lambda := \mathbb{R}$ is the real line and when the maps $\lambda \mapsto \mathrm{Graph}(F(\lambda,\cdot))$, $\lambda \mapsto \mathrm{Graph}(K(\lambda))$ and $\lambda \mapsto \mathrm{Graph}(C(\lambda))$ are not decreasing. Then the inverse $\mathbb{V}^{-1}$ associates with any $\lambda \in \mathbb{R}$ the lower level set*

$$\mathbb{V}^{-1}(\lambda) \;=\; \{x \in X \;\; \textit{such that}\, \nu(x) \leq \lambda\}$$

of the function $x \rightsquigarrow \nu(x)$ defined by

$$\nu(x) \;:=\; \inf_{(\lambda,x)\in \mathrm{Viab}_{(3.14)}(\mathcal{K},\mathcal{C})} \lambda \tag{3.15}$$

If the environment and the target do not depend upon the parameter λ, the inverse $\mathbb{I}^{-1}$ associates with any with any $\lambda \in \mathbb{R}$ the upper level set

$$\mathbb{I}^{-1}(\lambda) \;=\; \{x \in X \;\; \textit{such that}\, \lambda \leq \upsilon(x)\}$$

of the function $x \rightsquigarrow \upsilon(x)$ defined by

$$\upsilon(x) \;:=\; \sup_{(\lambda,x)\in \mathrm{Inv}_{(3.14)}(K\times\mathbb{R}_+,C\times\mathbb{R}_+)} \lambda \tag{3.16}$$

The function ν is the solution to the Hamilton-Jacobi-Bellman partial differential equation

$$\forall\, x, \quad \inf_{v\in F(\nu(x),x)} \left\langle \frac{\partial \nu(x)}{\partial x}, v \right\rangle \;=\; 0$$

in the Frankowska sense

$$\begin{cases} (i) & \inf_{v\in F(\nu(x),x)} D_{\uparrow}\nu(x)(v) \;\leq\; 0 \\ (ii) & \sup_{v\in -F(\nu(x),x)\cap DK(\lambda,x)(0)} D_{\uparrow}\nu(x)(v) \;\leq\; 0 \end{cases}$$

3.6.3 Crück's Example

Let us consider the case when $K(\lambda) := K \ominus \lambda B$, where B is a closed convex set containing 0 (a ball, for instance, or a "structuring element", as in mathematical morphology), where

$$K \ominus \lambda B \;:=\; \bigcap_{b\in B} (K - \lambda b)$$

is the *Minkowski difference* between K and λB.

To say that $x \in \mathrm{Viab}_F(K \ominus \lambda B, C) \backslash C$ means that there exists an evolution $x(\cdot) \in \mathcal{S}_F(x)$ to $x'(t) \in F(x(t))$ viable in $K \ominus \lambda B$ until it reaches the target C, if ever, i.e., such that

$$\forall\, t \geq 0, \;\; \forall\, b \in B, \;\; x(t) + \lambda b \;\in\; K$$

Since $\lambda \mapsto K \ominus \lambda B$ is non increasing, then the function $x \rightsquigarrow \lambda^\sharp(x)$ defined by

$$\lambda^\sharp(x) \;:=\; \sup\left\{\lambda \geq 0 \;\text{ such that } x \;\in\; \mathrm{Viab}_F(K \ominus \lambda B, C)\right\} \tag{3.17}$$

assigns to every $x \in K$ the size λ of a king of "guaranteed envelope" around at least one evolution starting at x.

Setting $\mathcal{K} := \mathrm{Graph}(\lambda \mapsto K \ominus \lambda B)$ and $\mathcal{C} := \mathbb{R}_+ \times C$ and assuming that F does not depend on λ (or that $\lambda \mapsto \mathrm{Graph}(F(\lambda, \cdot))$, we obtain the viability characterization of this function:

$$\lambda^\sharp(x) \;=\; \sup_{(\lambda,x) \in \mathrm{Viab}_{(3.14)}(\mathcal{K},\mathcal{C})} \lambda \tag{3.18}$$

which can be computed.

The function $\lambda^\sharp$ is the solution to the Hamilton-Jacobi-Bellman partial differential equation

$$\forall\, x, \;\; \inf_{v \in F(x)} \left\langle \frac{\partial \lambda^\sharp(x)}{\partial x}, v \right\rangle \;=\; 0$$

in the Frankowska sense

$$\begin{cases} (i) & \sup_{v \in F(x)} D_\downarrow \lambda^\sharp(x)(v) \;\geq\; 0 \\ (ii) & \inf_{v \in -F(x) \cap DK(\lambda,x)(0)} D_\downarrow \lambda^\sharp(x)(v) \;\geq\; 0 \end{cases}$$

In the case when $F(x) := f(x, U(x))$ is a control system, the regulation map R is defined by

$$R(x) \;\; \left\{u \in U(x) \;\text{ such that } D_\downarrow \lambda^\sharp(x)(f(x,u)) \;\geq\; 0\right\}$$

Viable evolutions are governed by the control system

$$\begin{cases} (i) & \lambda'(t) \;=\; 0 \\ (ii) & x'(t) \;\in\; f(x(t), u(t)) \\ & \text{where } u(t) \;\in\; R(x(t)) \end{cases} \tag{3.19}$$

If the system is Lipschitz, Quincampoix's Barrier Theorem implies that whenever $x \in \mathrm{Int}(K \ominus \lambda^{\sharp}(x)B)$, then $\lambda^{\sharp}(x(t)) = \lambda^{\sharp}(x)$ as long as $x(t) \in \mathrm{Int}(K \ominus \lambda^{\sharp}(x)B)$. After, we only know that $\lambda^{\sharp}(x(t)) \leq \lambda^{\sharp}(x)$.

3.7 Guaranteed Capture Basins under Dynamical Games

We summarize the main results on guaranteed viability/capturability of a target under dynamical games that we need to prove the results announced in the preceding section.

We denote by X, $\mathcal{U}$ and $\mathcal{V}$ three finite dimensional vector spaces, and we introduce a single-valued map $f : X \times \mathcal{U} \times \mathcal{V} \rightsquigarrow X$, a cybernetic set-valued map $U : X \rightsquigarrow \mathcal{U}$ and a tychastic set-valued map $V : X \rightsquigarrow \mathcal{V}$.

We consider a dynamical game described by

$$\left\{ \begin{array}{lrcl} (i) & x'(t) & = & f(x(t), u(t), v(t)) \\ (ii) & u(t) & \in & U(x(t)) \\ (iii) & v(t) & \in & V(x(t)) \end{array} \right. \tag{3.20}$$

which is, so to speak, a control system regulated by two parameters, $u(t)$ and $v(t)$, the first one regarded as a regulating parameter, controlled by a player, the second one regarded as a perturbation, or a disturbance, or a tyche, chosen in a unknown way by "Nature".

We introduce a class $\widetilde{\mathcal{P}}$ of continuous selections $x \mapsto \widetilde{u}(x) \in U(x)$, that are used as feedbacks or strategies by the player controlling the parameters u.

We associate with such a feedback $\widetilde{u}(x) \in U(x)$ the set $\mathcal{C}_{\widetilde{u}}(x)$ of solutions $(x(\cdot), v(\cdot)) \in \mathcal{C}(0, \infty; X) \times L^1(0, \infty; \mathcal{U})$ to the parameterized system

$$\left\{ \begin{array}{lrcl} (i) & x'(t) & = & f(x(t), \widetilde{u}(x(t)), v(t)) \\ (ii) & v(t) & \in & V(x(t)) \end{array} \right. \tag{3.21}$$

starting at x.

We may identify the above dynamical game with the set-valued map $(x, \widetilde{u}) \rightsquigarrow \mathcal{C}_{\widetilde{u}}(x)$, that we regard as an evolutionary game.

Definition 3.7.1. *Let $C \subset K \subset X$ be two subsets, C being regarded as a target, K as a constrained set.*

We denote by $\mathrm{Abs}_{\widetilde{u}}(K, C)$ the invariance-absorption basin of C in K, subset of initial states $x_0 \in K$ such that C is reached in finite time before possibly leaving K by all solutions to (3.21) starting at x_0.

The subset

$$[\mathrm{Capt}_P\mathrm{Abs}_V](K,C) \ := \ \bigcup_{\widetilde{u}\in\widetilde{\mathcal{P}}} \mathrm{Abs}_{\widetilde{u}}(K,C)$$

of elements $x \in K$ such that there exists a feedback $\widetilde{u} \in \widetilde{\mathcal{P}}$ such that for every solutions $(x(\cdot), v(\cdot)) \in \mathcal{C}_{\widetilde{u}}(x)$, there exists $t^* \in \mathbf{R}_+$ satisfying the viability/capturability conditions

$$\left\{ \begin{array}{lll} (i) & \forall\, t \in [0,t^*], & x(t) \in K \\ (ii) & & x(t^*) \ \in \ C \end{array} \right.$$

is called the guaranteed viable-capture basin of a target under the evolutionary game $(x,\widetilde{u}) \rightsquigarrow \mathcal{C}_{\widetilde{u}}(x)$ *defined on $X \times \widetilde{\mathcal{P}}$ (that, naturally, depends upon the choice of the family $\widetilde{\mathcal{P}}$ of feedbacks).*

Chapter 4

Dynamic Evaluation and Management of Portfolios

In this chapter devoted to finance, we shall depart of the notations used in these lecture notes to adopt notations familiar in the finance literature.

4.1 Description of the Model

4.1.1 State, regulatory and tychastic variables

We denote by

1. $i = 0, 1, \ldots, n$ assets ($i = 0$ denoting the non risky asset),
2. T the exercice time, and, at each running date t, $0 \leq t \leq T$, $T - t$ denoting "time to maturity".

The variables of the financial systems considered in this study are

1. the "state 'variables" of the system made of
   - the prices of the assets $S(t) := (S_0(t), S_1(t), \ldots S_n(t))$ ($S_0(t)$ being the price of the non risky asset, and $(S_1(t), \ldots S_n(t))$ the prices of the risky assets),
   - the number of shares of the assets making up the portfolio $P(t) := (P_0(t), P_1(t), \ldots, P_n(t))$,

- the value (capital) $W(t) := P_0(t)S_0(t) + \sum_{i=1}^{n} P_i(t)S_i(t)$ of the portfolio, where $P_0(t)S_0(t)$ is the "*liquid component*" of the portfolio and where the value $E(t) := \sum_{i=1}^{n} P_i(t)S_i(t)$ of the risky component of the value of the portfolio is the "*exposure*" of the portfolio.

2. the "controls", which are the *transactions* of the risky assets $P'(t) := (P_0'(t), P_1'(t), \ldots, P_n'(t))$, described by the time derivatives or the number of shares,
3. the "tyches", which are the returns $R(t) := (R_0(t), R_1(t), \ldots, R_n(t))$, where

$$\forall\, t \geq 0, \quad R_i(t) \ := \ \frac{S_i'(t)}{S_i(t)} \ = \ \frac{d\log(S_i(t))}{dt} \text{ if } S_i(t) \ > \ 0$$

of the prices of the assets. Tyches range over a *tychastic set* (that could be itself a fuzzy set).

4.1.2 The viability constraints

Viability theory deals with the problems of evolution under viability constraints bearing on state, regulatory and tychastic variables:

Financial Constraints on state variables

1. Constraints on prices
$$\forall\, t \ \in \ [0,T], \ \ S(t) \ \in \ \mathbb{S}(t)$$
2. Constraints on the shares des portfolios (liquidity constraints)
$$\forall\, t \ \in \ [0,T], \ \ P(t) \ \in \ \mathbb{P}(t, S(t), W(t))$$
3. Constraints on the value of the portfolio describing guarantees by a *threshold or floor function* $\mathbf{b}(t,S)$
$$\forall\, t \ \in \ [0,T], \ \ W(t) \ \geq \ \mathbf{b}(t, S(t))$$
4. *Cash-Flows* are described by dates T_k payment functions $(S,W) \mapsto \pi(T_k, S, W)$ subtracting to the capital, at dates T_k, amounts $\pi(T_k, S(T_k), W(T_k))$ associated with functions $t \mapsto S(t)$ and $t \mapsto W(t)$.

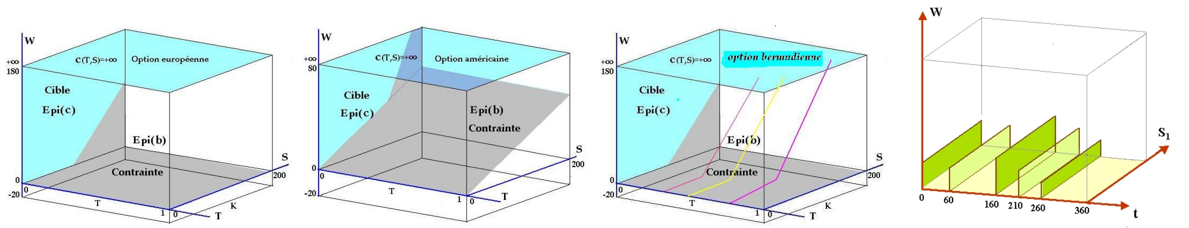


Figure 4.1.1. (Examples of threshold functions) *From left to right, threshold functions for European, American, Bermudian options and cash flows.* Financial Rules *involve constraints requiring that at each instant, the value of the portfolio must be larger than or equal to a threshold function depending of the time at maturity, the price and the number of shares of the portfolio.*

1. *For portfolios replicating European options, the threshold is equal to zero before the exercise time and to the contingent function at exercice time,*

2. *For portfolios replicating a type of American options, the threshold is equal to a given percentage of the price before the exercise time and to the supremum of this function and the contingent function at exercice time,*

3. *For portfolios replicating Bermudian options, the threshold is equal to zero except at a finite set of dates when it is a contingent function,*

4. *The threshold fonction can also describe a cash flow that has to be satisfied at each instant.*

No restriction is made in the choice of the threshold function which defines the "financial rules".

Financial Constraints on tychastic variables The returns must obey "tychastic constraints"

$$\forall\, t \;\in\; [0,T], \;\; R(t) \;\in\; \mathbb{R}(t, S(t), P(t), W(t))$$

where the set-valued map $\mathbb{R}(t, S(t), P(t), W(t))$ is called the *tychastic map*.

We provide below an example of a tychastic map in the case of one risky asset ($n = 1$): The interest rates of the non risky asset $R_0(t)$ are given and the returns $R(t) := R_1(t)$ of the risky asset satisfy

$$\forall\, t \;\in\; [0,T], \;\; R(t) \;\in\; \mathbb{R}(t, S(t), P(t), W(t)) \;:=\; [R^\flat(t), R^\sharp(t)]$$

and in particular, when

$$\forall\, t \in [0,T], \quad R(t) \in \mathbb{R}(t, S(t), P(t), W(t)) := [R - \nu(t), R + \nu(t)]$$

where the function $\nu(\cdot)$ is the tychastic versatility threshold.

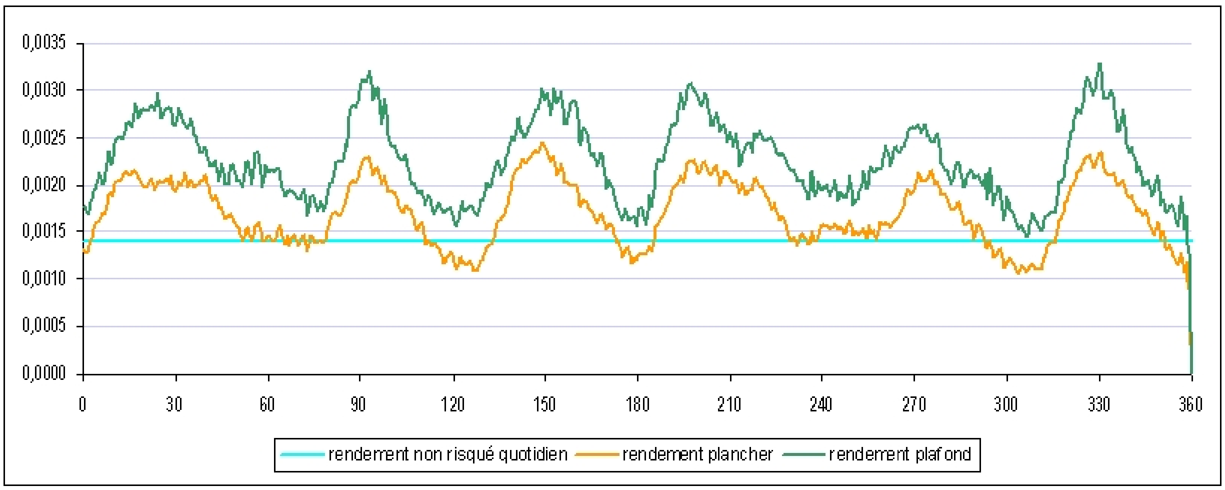


Figure 4.1.2. Representation of Tychastic Uncertainty *The picture displays the daily interest rate of the non-risky asset (light gray line), of the daily floor (dark gray) and ceiling (black) returns of the risky asset describing the tychastic scenario.*

Financial Constraints on regulatory variables

Constraints on transactions are described by subsets $\mathbb{F}(t, S, P, W)$:

$$P'(t) \in \mathbb{F}(t, S(t), P(t), W(t))$$

The two main examples of constraints on transactions are

1. *Trading Constraints*, of the form $|P_i'(t)| \leq \gamma_i(t)$, $i = 1, \ldots, n$, $0 \leq \gamma_i(t) \leq +\infty$, the case $\gamma_i(t) = 0$ translating an impossibility of trading at date t, the case $\gamma_i(t) = +\infty$ expressing the absence of trading constraints at this date,
2. *Transaction Costs*,

$$\sum_{i=0}^{n} P_i'(t)S_i(t) \;=\; -\delta(P'(t), P(t), S(t), W(t))$$

"Self-financed portfolios" are the special case when the transaction cost function does not involve transactions, such as

$$\sum_{i=0}^{n} P_i'(t)S_i(t) \;=\; 0 \text{ or, more generally } \sum_{i=0}^{n} P_i'(t)S_i(t) \;=\; \varphi(t, S(t))W(t)$$

This is an important case because the shares of the portfolio are no longer state variables, but controls (see section 4.2.5).

4.1.3 The dynamics

The state variables (S_i, P_i, W) must evolve in the time dependent constrained set $\mathcal{K}(t)$ defined by

$$\mathcal{K}(t) \;:=\; \{(S, P, W) \mid S \;\in\; \mathbb{S}(t),\; P \;\in\; \mathbb{P}(t, S, W) \;\&\; W \;\geq\; \mathbf{b}(t, S)\} \tag{4.1}$$

In order to define option contracts where the option is exercised at an opportune or propitious time $t^\star$, we introduce a time-dependent target $\mathcal{C}(t) \subset \mathcal{K}(t)$ and require that at time $t^\star$,

$$(S(t^\star), P(t^\star), W(t^\star)) \;\in\; \mathcal{C}(t^\star)$$

.

An example of target is associated with a "target function" $\mathbf{c}(t, S) \;\geq\; \mathbf{b}(t, S)$ in the following way:

$$\mathcal{C}(t) \;:=\; \{(S, P, W) \mid S \;\in\; \mathbb{S}(t),\; P \;\in\; \mathbb{P}(t, S, W) \;\&\; W \;\geq\; \mathbf{c}(t, S)\} \tag{4.2}$$

This means that the option is exercised at the first time $t^\star$ when

$$W(t^\star) \;\geq\; \mathbf{b}(t^\star, S(t^\star))$$

Other option contracts are obtained by taking $\mathbf{b}(t, S) = 0$ et $\mathbf{c}(t, S) = \max(S - K, 0)$: The option is exercised as soon as there exists a time $t^\star$ such that $W(t^\star) \geq \max(S(t^\star) - K, 0)$.

Some contracts may involve as target functions the valuation function of other contracts, as in "barrier options".

The dynamical system governing the evolutions of the state variables: For $i = 0, 1, \ldots, n$,

$$\left\{ \begin{array}{lrcl} (i) & S_i'(t) & = & R_i(t)S_i(t),\ i = 0, \ldots, n, \text{ where } R(t) \in \mathbb{R}(t, S(t), P(t), W(t)) \\ (ii) & P_i'(t) & = & u_i(t),\ i = 0, \ldots, n, \text{ where } u(t) \in \mathbb{F}(t, S(t), P(t), W(t)) \\ (iii) & W'(t) & = & R_0(t)W(t) + \sum_{i=1}^{n} P_i(t)S_i(t)(R_i(t) - R_0(t)) + \sum_{i=0}^{n} u_i(t)S_i(t) \end{array} \right. \tag{4.3}$$

parameterized by the controls $u_i := P_i'$, which are the transactions, and the tyches R_i, which are the rates of the risky assets. This is a "*tychastic control system*" or a differential game against nature.

4.1.4 Cash-Flows

Impulse dynamics are hybrid dynamics introducing discontinuities in the evolutions when the capital hits the threshold function. There is a general theory for dealing with these questions with viability techniques which can be applied to financial models.

Cash-Flows are defined by finite sequences of dates $0 =: T_0 < T_1 < T_2 < \ldots < T_{N-1} < T_N =: T$ at which payments $\pi(T_i, S, W, P)$ must be made: We set $W(T_i^-) := \lim_{t \leq T_i,\, t \to -T_i} W(t)$. At this date, the payment must be done in an impulsive way: The new capital $W(T_i)$ at date T_i becomes:

$$\forall\, i = 1, \ldots, N,\ \ W(T_i) \ = \ W(T_i^-) - \pi(T_i, S(T_i), W(T_i))$$

A necessary condition is that at date T_i, the capital $W(^-T_i)$ satisfies

$$\forall\, i = 1, \ldots, N,\ \ W(T_i^-) \ \ \geq \mathbf{b}(T_i, S(T_i)) + \pi(T_i, S(T_i), W(T_i))$$

4.2 Guaranteed Capture Basins and Viability Kernels

4.2.1 Definition

Definition 4.2.1. (Guaranteed Viability Kernel) *Given an exercice time T, a time-dependent constrained sets $\mathcal{K}(t)$ defined by (4.1) and a time-dependent target $\mathcal{C}(t) \subset \mathcal{K}(t)$*

defined by (4.2), its time-dependent guaranteed capture basin

$$\mathcal{V}(t) \ := \ \text{GuarCapt}_{(4.3)}(\mathcal{K}, \mathcal{C})(t)$$

under the tychastic control system (4.3) is the tube $\tau \rightsquigarrow \mathcal{V}(\tau)$, $\tau \in [0,T]$, *made of elements* $(S,P,W) \in \mathcal{V}(\tau)$ *for which there exists a feedback map* $\mathbb{G}(t,S,P,W) \in \mathbb{F}(t,S,P,W)$ *such that, for any selection of returns* $R(t) \in \mathbb{R}(t,S(t),P(t),W(t))$, *there exists a time* $t^\star \in [0,T]$ *such that the evolution of* $(S(t),P(t),W(t))$ *governed by the system of differential equations*

$$\left\{ \begin{array}{lll} (i) & S_i'(t) \ = \ R_i(t)S_i(t), \ \ i = 0,\ldots,n, \\ (ii) & P'(t) \ = \ \mathbb{G}(t,S(t),P(t),W(t)) \\ (iii) & W'(t) \ = \ R_0(t)W(t) + \displaystyle\sum_{i=1}^{n} P_i(t)S_i(t)(R_i(t) - R_0(t)) + \sum_{i=0}^{n} \mathbb{G}_i(t,S(t),P(t),W(t))S_i(t) \end{array} \right. \tag{4.4}$$

and starting at time τ *from* (S,P,W) *reaches the target at time* $t^\star$ *in the sense that*

$$(S(t^\star),P(t^\star),W(t^\star)) \ \in \ \mathcal{C}(t^\star)$$

and is meanwhile viable in $\mathcal{K}(t)$ *in the sense that*

$$\forall \, t \in [\tau, t^\star], \ \ (S(t),P(t),W(t)) \ \in \ \mathcal{K}(t)$$

Whenever the time-dependent target $\mathcal{C}(t)$ is equal to

$$\mathcal{C}_{\mathcal{K}}(t) \ := \ \emptyset \text{ if } 0 \leq t < T \text{ and } \mathcal{C}_{\mathcal{K}}(T) \ := \ \mathcal{K}(T)$$

then the guaranteed capture basin

$$\text{GuarViab}_{(4.3)}(\mathcal{K})(t) \ := \ \text{GuarCapt}_{(4.3)}(\mathcal{K}, \mathcal{C}_{\mathcal{K}})(t)$$

is called the *time-dependent guaranteed viability kernel*

$$\mathcal{V}(t) \ := \ \text{GuarViab}_{(4.3)}(\mathcal{K})(t)$$

of the time-dependent environment $\mathcal{K})(t)$ under the tychastic control system (4.3). In this case, *the time* $t^\star = T$ *is equal to the exercice time* T.

The introduction of non trivial targets allows us to cover many other option contracts which are exercised as soon as the state $(S(t^\star),P(t^\star),W(t^\star)) \ = \ \mathcal{C}(t^\star)$.

We restrict our attention to the links between the concepts of guaranteed of capture basin and viability kernel in the particular case of time-dependent constrained sets $\mathcal{K}(t)$ defined by (4.1), time-dependent target $\mathcal{C}(t) \subset \mathcal{K}(t)$ defined by (4.2) and tychastic control system (4.3).

4.2.2 Valuation Function and The Transaction Rule

Knowing the guaranteed viability kernel, we can deduce easily the answers to the problem of the evaluation of the capital and the management of the shares making up the portfolio in the following way:

Theorem 4.2.2. (Valuation and Management of the portfolio) *Given an exercice time T and the time-dependent constrained sets $\mathcal{K}(t)$ defined by (4.1), the* time-dependent guaranteed viability kernel

$$\mathcal{V}(t) \ := \ \mathrm{GuarViab}_{(4.3)}(\mathcal{K})(t)$$

under the tychastic control system (4.3) provides

1. *the* initial capital
$$\mathbb{W}(0,S,P) \ := \inf_{(S,P,W)\in\mathcal{V}(0)} W$$

2. *the* initial portfolio $\mathbb{Q}(0,S)$*, which minimizes the function $P \mapsto \mathbb{W}(0,S,P)$ over the subset $\mathbb{P}(0,S,\mathbb{W}(0,S,P))$, i.e., a fixed point of the problem*
$$\mathbb{W}(0,S,\mathbb{Q}(0,S)) \ = \ \mathbb{V}(0,S) \ := \inf_{P\in\mathbb{P}(0,S,\mathbb{W}(0,S,\mathbb{Q}(0,S)))} \mathbb{W}(0,S,P)$$
(whenever the constraints on the shares depend upon W)

3. *the* transaction rule
$$P'(t) \ = \ \mathbb{G}(t,S(t),P(t),W(t))$$
defined by the feedback involved in the definition of the time-dependent guaranteed viability kernel.

Consequently, for any evolution of the prices $S(t)\in\mathbb{S}(t)$, the shares $P(t)$ and the capital $W(t)$ evolve according the system of differential equations

$$\left\{\begin{array}{lll} (i) & P'(t) \ = \ \mathbb{G}(t,S(t),P(t),W(t)) \\ (ii) & W'(t) \ = \ R_0(t)W(t) + \displaystyle\sum_{i=1}^{n} P_i(t)S_i(t)(R_i(t)-R_0(t)) + \sum_{i=0}^{n} \mathbb{G}_i(t,S(t),P(t),W(t))S_i(t) \end{array}\right. \tag{4.5}$$

starting from the initial portfolio $\mathbb{Q}(0,S)$ and the initial capital $\mathbb{V}(0,S)=\mathbb{W}(0,S,\mathbb{Q}(0,S))$.

Viability theory studies in depth the properties of the time-dependent viability kernels under tychastic control problems. The key point is that there is an algorithm computing the time-dependent guaranteed viability kernel when time, state, regulatory and tychastic variables are discetized. Difficult convergence theorems guarantee the convergence under adequate assumptions.

4.2.3 Options with Trading Constraints

Consider the case when there exists only one risky asset ($n = 1$). The *constraints* bear on

1. prices of the risky asset:

$$\forall\, t \;\in\; [0, T],\;\; S(t) \;\in\; [S^{\flat}(t), S^{\sharp}(t)]$$

   where $S^{\flat}(t) \geq 0$,

2. the shares of the risky asset (liquidity constraints)

$$\forall\, t \;\in\; [0, T],\;\; P(t) \;\in\; \left[P^{\flat}(t), \min\left(P^{\sharp}(t), \frac{W}{S}\right)\right]$$

   (which imply that $P_0(t) \geq 0$ whenever $P^{\flat}(t) \geq 0$),

3. the values of the portfolio, described by a threshold function $\mathbf{b}(t, S)$

$$\forall\, t \;\in\; [0, T],\;\; W(t) \;\geq\; \mathbf{b}(t, S(t))$$

   where $\mathbf{b}$ may be discontinuous (but at least lower semicontinuous)

4. *trading constraints*:

$$\forall\, t \in [0, T],\;\; |P'(t)| \;\leq\; \gamma(t)$$

   where γ may be discontinuous (but at least upper semicontinuous). This is the case for treating "*rebalancing*" constraints, when $\gamma(t) = 0$ except at discrete times when transactions are allowed to be made.

5. a "tychastic" translation of uncertainty:

$$\forall\, t \in [0, T],\;\; r(t) - \nu(t) \;\leq\; R(t) \;\leq\; r(t) + \nu(t)$$

   (where the tychastic versatility threshold function $\nu(\cdot)$ is assumed to be Lipschitz).

We denote by $\mathcal{K}(W)$ the subset of triples (t, S, P) such that $0 \leq t \leq T$, $S^\flat(t) \leq S \leq S^\sharp(t)$, $P^\flat(t) \leq P \leq \min(S^\sharp(t), \frac{W}{S})$ and $W \geq \mathbf{b}(t, S)$ and by $\mathcal{C}(t)$ the subset of elements of $\mathcal{K}$ such that $W \geq \mathbf{c}(t, S)$ where $\mathbf{c}(t, S) = +\infty$ if $t < T$ and $\mathbf{b}(T, S) = \mathbf{c}(T, S)$.

One can prove that the function $(t, S, P) \mapsto \mathbb{W}(t, S, P)$ is the unique solution (in an adequate generalized sense) of a free boundary problem for the following (nonlinear) partial differential equation with discontinuous coefficients: for all $(t, S, P) \in \mathcal{K}(W)$,

$$\left\{ \begin{array}{l} \dfrac{\partial \mathbb{W}}{\partial t} + \dfrac{\partial \mathbb{W}}{\partial S} r(t) S + \nu(t) S \left| \dfrac{\partial \mathbb{W}}{\partial S} - P \right| - \gamma(t) \left| \dfrac{\partial \mathbb{W}}{\partial P} - S \right| \\ = \ r_0 \mathbb{W} + PS(r(t) - r_0) \end{array} \right. \tag{4.6}$$

satisfying the final condition $\mathbb{W}(T, S, P) = \mathbf{c}(T, S)$. *This is the tychastic version of the Black and Scholes equation adapted to this problem.*

Observe (informally) that if the versatility $\nu(t) = +\infty$ is infinite and if there is no constraint on the number of shares, then $P = \dfrac{\partial \mathbb{W}}{\partial S}$, which is the famous Δ-hedging rule. If there is no restriction on trading, then we have $S = \dfrac{\partial \mathbb{W}}{\partial P}$.

This is a highly non linear problem because not only it involves a first order nonlinear partial differential equation with discontinuous coefficients (instead of a second linear one as the Black and Scholes) but above all, because the subset $\mathcal{K}(W)$ on which it is defined ... depends upon the solution of this equation.

The transaction rule is given by

$$P'(t) \ = \ -\gamma(t) \frac{\frac{\partial \mathbb{W}}{\partial P} - S}{\left| \frac{\partial \mathbb{W}}{\partial P} - S \right|} \tag{4.7}$$

4.2.4 Example: European Options With Transaction Costs

The tychastic approach allows us to treat transaction costs, whereas the stochastic one raises many difficulties (see a paper untitled *There is no trivial hedging for option pricing with transaction costs* by Soner H.M., Shreve S.E. & Cvitanic.

We assume that $S(t) \geq 0$ and that $P(t) \in [0, P^\sharp]$. The threshold function for the European option is defined by

$$\mathbf{b}(t, S) \ = \left\{ \begin{array}{l} 0 \text{ if } t < T \\ \max(S - K, 0) \text{ if } t = T \end{array} \right.$$

We consider two types of constraints on the transactions:

- *Trading Constraints*:

$$\forall\, t \geq 0, \;\; |P'(t)| \;\leq\; \gamma(t)$$

- *Transaction costs:*

$$P'(t)S(t) \;=\; -\delta|P'(t)|S(t)$$

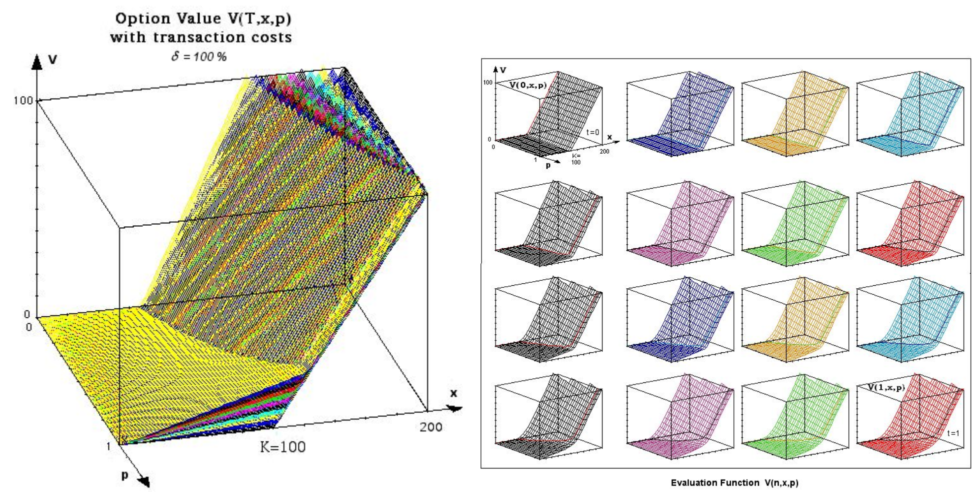


Figure 4.2.3. (Valuation function *The figure displays the valuation function* $\mathbb{W}(0, S, P))$ *for several values of* δ *and a fixed exercice time (left) and the the value fonctions for a fixed cost* δ *and several exercice times.*

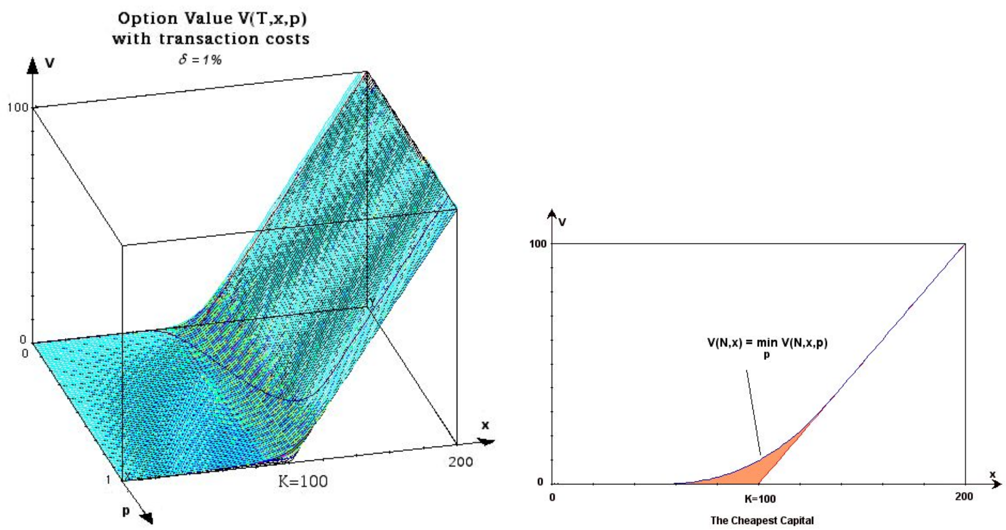
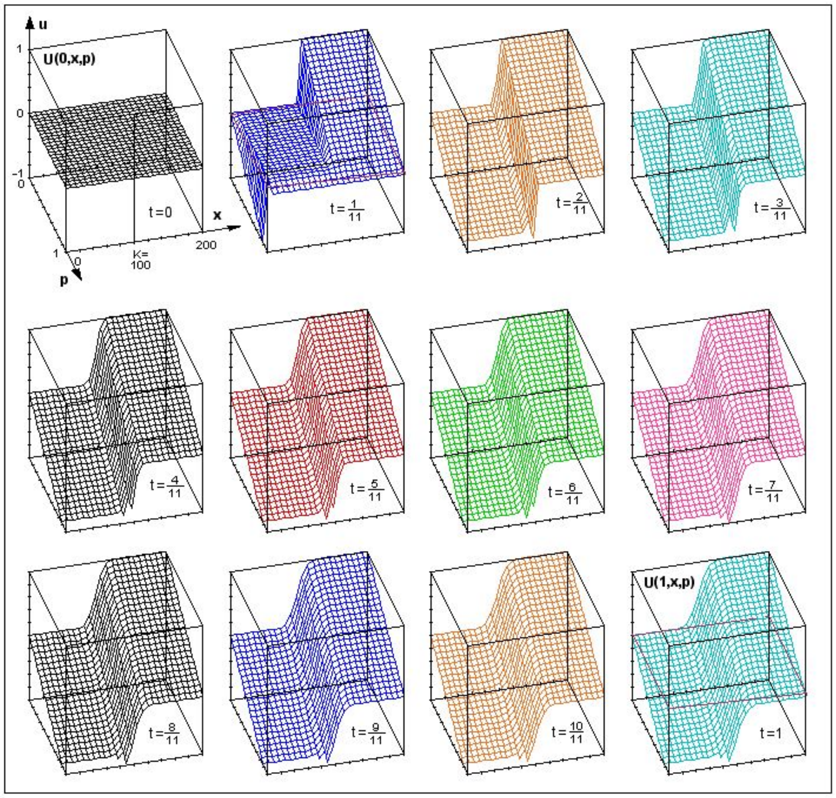


Figure 4.2.4. (Valuation Fonctions) *This figures displays the valuation function* $\mathbb{V}(0,S) := \inf_{P\in\mathbb{P}(t)} \mathbb{W}(0,S,P)$ *for a given exercice time* T *in the graph of* $\mathbb{W}(0,S,P)$ *(left), the graph of the function* $S \mapsto \mathbb{V}(0,S)$ *(right).*



Optimal Buy & Sell strategy **U(t,x,p)**

t : time left until exercice time

Figure 4.2.5. (Transaction rules) *This figure displays the graph of the transaction rule* $(S, P) \mapsto \mathbb{G}(t, S, P)$ *for several times to maturity. When the time to maturity is equal to* 0, $\mathbb{G}(0, S, P) = 0$, *because there is no transaction at exercice time. The transactions are negative far below the exercice time and positive far above, a quite intuitive statement.*

4.2.5 Particular Case of Self-Financing Portfolios

In the case of self-financing portfolios where

$$\sum_{i=0}^{n} P_i'(t)S_i(t) \;=\; \varphi(t, S(t))W(t)$$

the transactions disappear in the tychastic control system (4.3), which boils down to the simplified tychastic control system

$$\left\{\begin{array}{ll} (i) & S_i'(t) \;=\; R_i(t)S_i(t) \text{ where } R(t) \;\in\; \mathbb{R}(t,S(t),P(t),W(t)) \\ (ii) & W'(t) \;=\; (R_0(t)+\varphi(t,S(t)))W(t)+\displaystyle\sum_{i=1}^{n} P_i(t)S_i(t)(R_i(t)-R_0(t)) \\ & \quad \text{where } P(t) \;\in\; \mathbb{P}(t,S(t),W(t)) \end{array}\right. \tag{4.8}$$

where the tyches are still the returns and the controls the numbers of shares instead of their transactions.

The state variables (S,P,W) must evolve in the time dependent constrained set $\mathcal{K}(t)$ defined by

$$\mathcal{K}(t) \;:=\; \{(S,W) \mid S \;\in\; \mathbb{S}(t) \;\&\; W \;\geq\; \mathbf{b}(t,S)\} \tag{4.9}$$

Definition 4.2.6. (Guaranteed Viability Kernel) *Given an exercice time T and the time-dependent constrained sets $\mathcal{K}(t)$ defined by (4.9), its* time-dependent guaranteed viability kernel

$$\mathcal{V}(t) \;:=\; \mathrm{GuarViab}_{(4.8)}(\mathcal{K})(t)$$

under the tychastic control system (4.8) is the tube $\tau \rightsquigarrow \mathcal{V}(\tau)$, $\tau \in [0,T]$, made of elements $(S,W) \in \mathcal{V}(\tau)$ for which there exists a feedback map $\mathbb{G}(t,S,W) \in \mathbb{P}(t,S,W)$ such that, for any selection of returns $R(t) \in \mathbb{R}(t,S(t),W(t))$, the evolution of $(S(t),W(t))$ governed by the system of differential equations

$$\left\{\begin{array}{ll} (i) & S_i'(t) \;=\; R_i(t)S_i(t), \;\; i=0,\ldots,n, \\ (ii) & W'(t) \;=\; (R_0(t)+\varphi(t,S(t)))W(t)+\displaystyle\sum_{i=1}^{n} \mathbb{G}_i(t,S(t),W(t))S_i(t)(R_i(t)-R_0(t)) \end{array}\right. \tag{4.10}$$

and starting at time τ from (S,W) is viable in $\mathcal{K}(t)$ in the sense that

$$\forall\, t \in [\tau,T], \;\; (S(t),W(t)) \;\in\; \mathcal{K}(t)$$

Knowing the guaranteed viability kernel, we derive:

Theorem 4.2.7. (Valuation and Management of the portfolio) *Given an exercice time T and the time-dependent constrained sets $\mathcal{K}(t)$ defined by (4.9), the* time-dependent guaranteed viability kernel

$$\mathcal{V}(t) \;:=\; \mathrm{GuarViab}_{(4.8)}(\mathcal{K})(t)$$

under the tychastic control system (4.8) provides at each instant t*,*

1. *the* capital
$$\forall\, t \in [0,T], \quad \mathbb{W}(t,S) \;:=\; \inf_{(S,W)\in\mathcal{V}(t)} W$$

2. *the* management rule
$$\mathbb{P}(t,S) \;=\; \mathbb{G}(t,S,\mathbb{W}(t,S))$$
defined by the feedback involved in the definition of the time-dependent guaranteed viability kernel.

Consequently, for any evolution of the prices $S(t) \in \mathbb{S}(t)$*, the shares and the capital are given by* $W(t) := \mathbb{W}(t,S(t))$ *and* $P(t) = \mathbb{P}(t,S(t))$*.*

The very same viability techniques allow to treat the "implied versatility" issue. Usually, it is assumed that the portfolio is self-financed. Consider the case of one risky asset. Given the classical contingent function $\max(0, S-K)$ where K is the striking price, an exercise time T and a *constant* tychastic threshold ν, one can associate with any (T,S,K,ν) the initial value $W := \Theta(T,S,K,\nu)$ of the portfolio such that there exists a feedback map $\mathbb{Q}(t,S,W,K,\nu) \in \mathbb{P}(S,W)$ such that, for any selection of returns $v(t) \in [-\nu,+\nu]$, the evolution of $(S(t),W(t))$ governed by the system of differential equations

$$\left\{\begin{array}{lrcl} (i) & S'(t) & = & r(t)S(t) + v(t)S(t) \\ (ii) & W'(t) & = & r_0 W(t) + P(t)S(t)(r - r_0 + v(t)) \text{ where } P(t) := \mathbb{Q}(t,S(t),W(t),K,\nu) \end{array}\right. \tag{4.11}$$

starting from (S,W) satisfy $W(t) \geq 0$ and

$$W(T) \;:=\; \Theta(T,S,K,\nu) \;\geq\; \max(0, S(T)-K)$$

The *implied versatility function* associates with any (T,S,K,W) the largest versatility threshold $\nu := \Lambda(T,S,K,W)$ under which

$$\left\{\begin{array}{ll} (i) & \forall\, W \geq 0, \;\; \Theta(T,S,K,\Lambda(T,S,K,W)) \leq W \\ (ii) & \forall\, \nu \geq 0, \;\; \Lambda(T,S,K,\Theta(T,S,K,\nu)) \;\geq\; \nu \end{array}\right. \tag{4.12}$$

These two functions can be characterized in terms of guaranteed viability kernels and computed by the Capture Basin Algorithm instead of inverting the function $\nu \mapsto \Theta(T,S,K,\nu)$ by standard inversion methods which do not take into account its viability property.

4.2.6 Cash-Flow (without Transaction Costs)

In this example, the constraints are $S(t) \geq 0$, $0 \leq P(t) \leq P^\sharp$, $W(t) \geq 0$ and the cash-flow is made of payments $\pi(T_i, S, W) := \pi_i$.

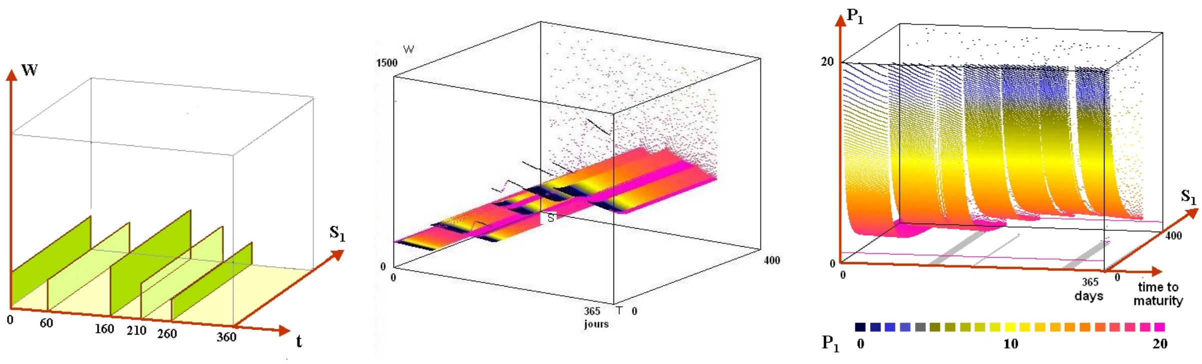


Figure 4.2.8. (Example of cash flows with constraints on the shares but without transaction constraints: Capital and Shares in terms of exercice time and prices) *Cash flow, capital and shares of the risky asset in terms of exercice time (abscissa) and price of the risky asset (ordinate)*

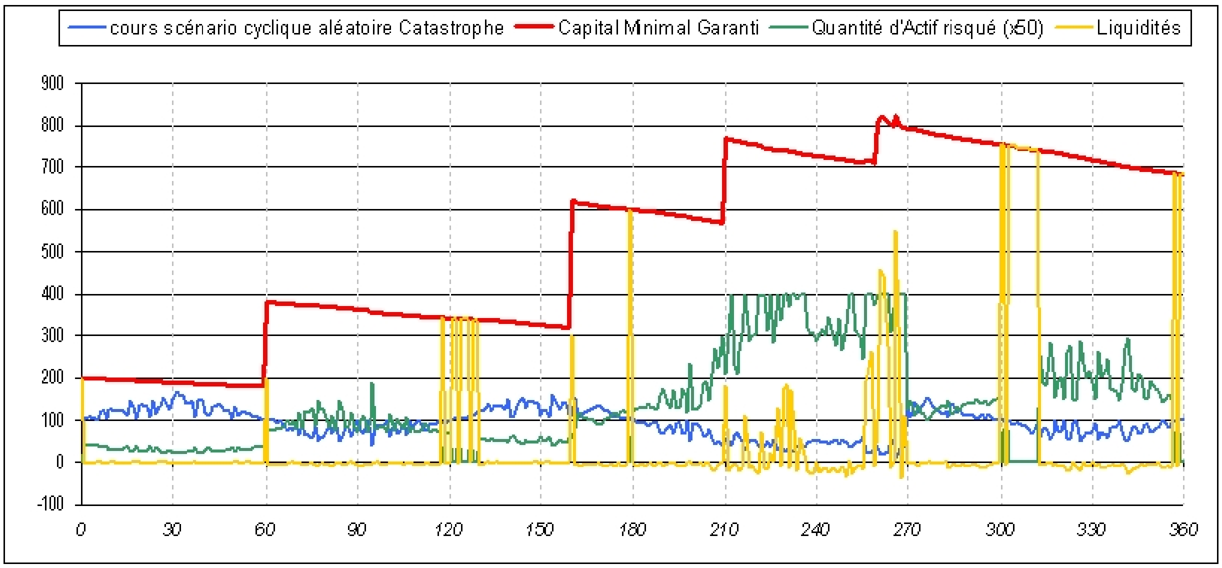


Figure 4.2.9. Guaranteed Evolution of Value and Shares (Scenario A) *The evolution of the price od the risky asset is simulated (dark gray curve). Note the drop of the prices. The picture displays the evolution of the associated value of the portfolio (in black), the number of shares of the risky asset (in gray) the value of the non-risky component of the portfolio in light gray.*

4.3 Options without transaction constraints

We require that at the *exercise time* T, the option is exercised. The threshold function for classical European, American and Bermudian options are

$$\mathbf{b}(t,S) \;=\; \begin{cases} (i) & 0 \text{ if } t < T \text{ and } \max(S-K,0) \text{ if } t = T \\ & \text{European Options} \\ (ii) & \max(S-K,0) \text{ if } t \leq T \\ & \text{American Options} \\ (iii) & aS \text{ if } t < T \text{ and } \max(S-K,aS) \text{ if } t = T, \;\; 0 < a \leq 1 \\ & \text{Quasi-American Options} \\ (iv) & 0 \text{ if } t \neq T_i \text{ and } \max(S-K_i,0) \text{ if } t = T_i, \; i = 1,\ldots,n \\ & \text{Bermudian Options} \end{cases} \tag{4.13}$$

4.3.1 European Options Without Transaction Costs

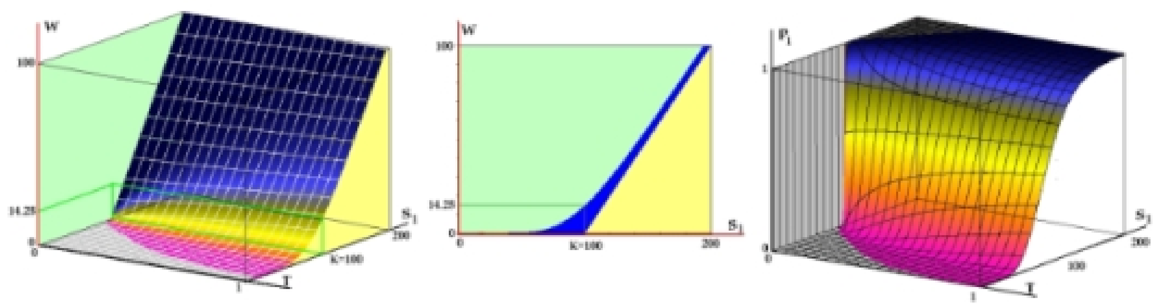


Figure 4.3.1. (European Option Without Transaction Costs) *This Figure displays the valuation function and the price function.*
Left and Right: Abscissas: Time to maturity, Ordinates: Prices of the Risky Asset.
Left : Price of the European Option, Right: Number of Shares.
Middle : For a fixed exercise time, Abscises: Prices of the Risky Asset. Ordinates: Price of the European Option.

Maturité	Volatilité	Black & Scholes	Bassin de Capture	Couverture π_0	Couverture π_1
1	10%	**6.72**	**6.82**	-64.06	0.7078
0.5	10%	**4.15**	**4.21**	-60.95	0.6507
0.1	10%	**1.52**	**1.53**	-55.47	0.5699
1	20%	**10.45**	**10.46**	-53.23	0.6368
0.5	20%	**6.89**	**6.89**	-52.91	0.5979
0.1	20%	**2.77**	**2.79**	-51.68	0.5446
1	25%	**12.27**	**12.34**	-50.41	0.6274
0.5	25%	**8.22**	**8.26**	-50.84	0.5910
0.1	25%	**3.39**	**3.42**	-50.72	0.5413
1	30%	**14.23**	**14.23**	-48.20	0.6242
0.5	30%	**9.65**	**9.64**	-49.24	0.5887
0.1	30%	**4.03**	**4.04**	-49.97	0.5401
1	50%	**21.73**	**21.76**	-41.69	0.6344
0.5	50%	**15.09**	**15.12**	-44.66	0.5977
0.1	50%	**6.53**	**6.54**	-47.87	0.5441

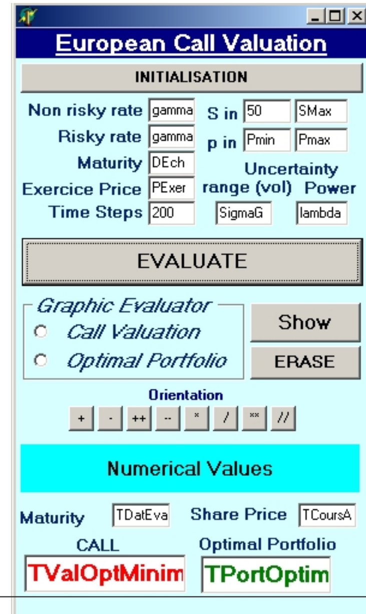


Figure 4.3.2. (Comparison of Algorithms)

Actually, there are two questions : The first one deals with the approximation of the Black and Scholes formula for continuous time by discrete time problems, and the second deals with the computation of the solution to this approximated discrete problem. This is for solving the discretized problem (both with respect to time and space variables) that the Capture Basin Algorithm is used. The other issue deals with the convergence of the solution to the discrete problems to the solution of the continuous time problem. It happens that the discretization of the stochastic problem and of the tychastic problems are quite the same, up to the replacement of the step size Δt in the tychastic discrete system by $\sqrt{\Delta t}$ in some terms of the discrete stochastic system, which provides the Cox, Ross and Rubinstein algorithm in he case of portfolios replicating European options. Hence, by modifying the discretization of the tychastic system by an adequate discretization of the stochastic system, the Viability Kernel Algorithm provides pricers, evaluation of the value of the portfolio and the regulation rule for both mathematical translations of uncertainty, the tychastic one allowing to take into account constraints on the versatility depending upon time, asset prices, and shares of the portfolios.

4.3.2 Other Options Without Transaction Costs

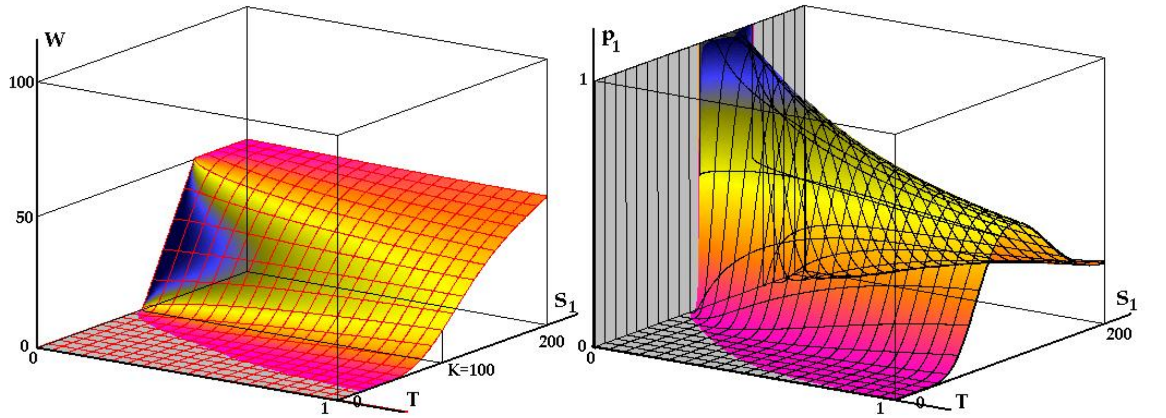


Figure 4.3.3. ("Capped" Options *Value and number of shares of risky assets.*

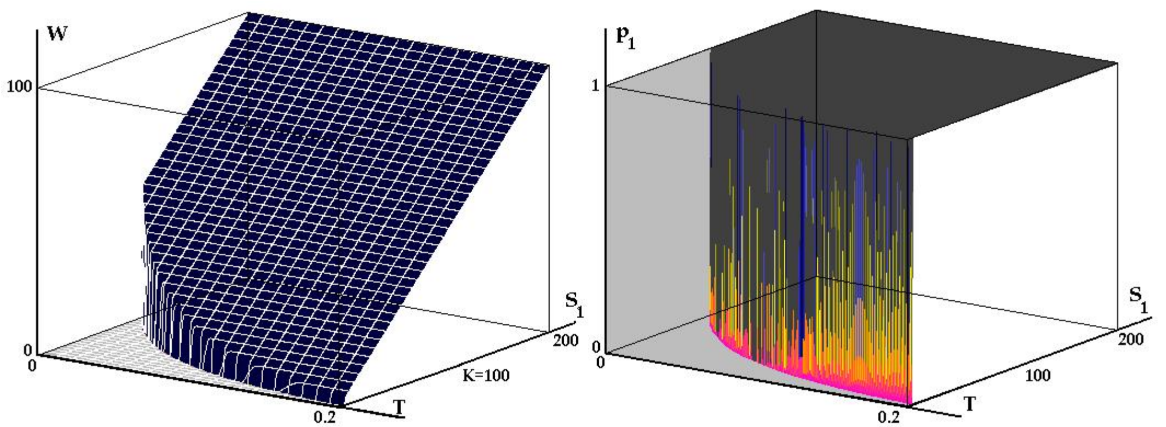


Figure 4.3.4. ("Asset or Nothing" Options) *Value and number of shares of risky assets.*

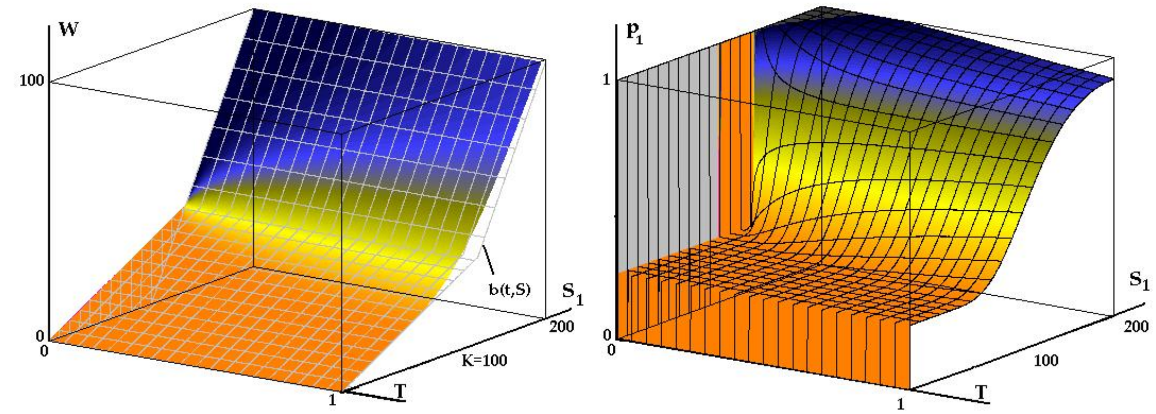


Figure 4.3.5. ("Non-Standard" American Options) *Value and number of shares of risky assets.*

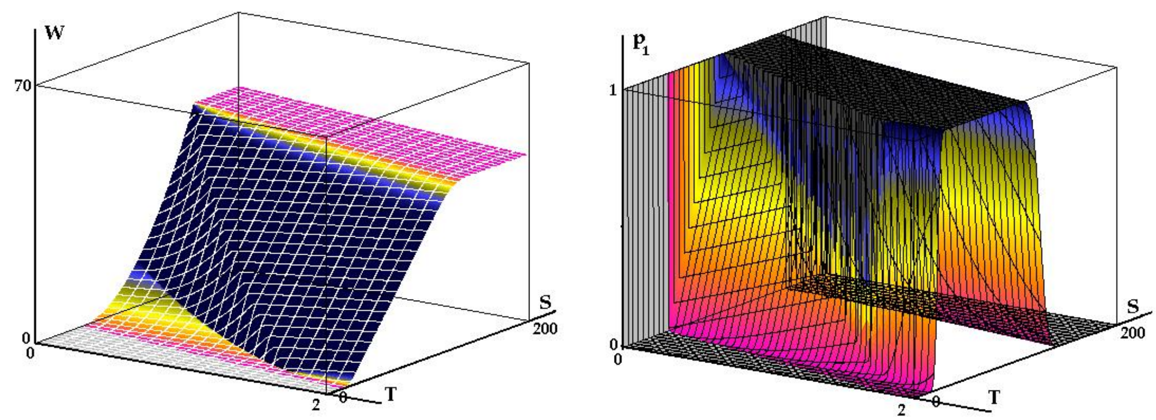


Figure 4.3.6. ("Non-Standard" Options) *Value and number of shares of risky assets under another tychastic dynamics without transactions costs. We take* $r(t,S) = \frac{\sqrt{S}}{1000}$, $\vartheta(t) = 0.3\frac{1}{0.01+t^2}$.

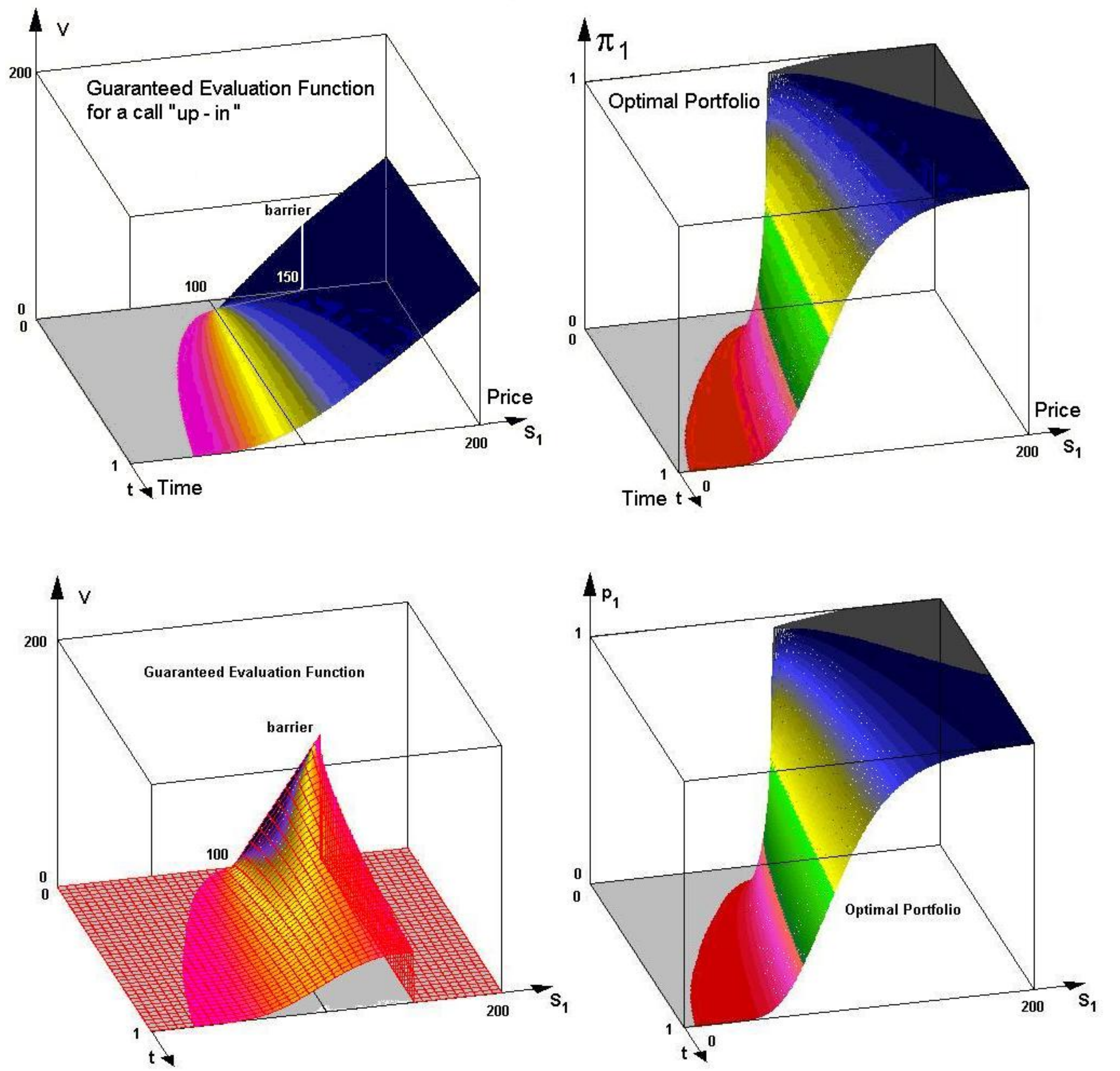


Figure 4.3.7. (European call with barrier "up in" and "up out") *Value and number of shares of risky assets.*

We observe a kind of stability of the shape of the valuation function in all these examples, but the nature of the management rule is a very sensitive to the change of contracts.

4.4 Viabilist Portfolio Insurance and CPPI

We introduce the following financial constraints on the exposure of a portfolio made of one risky asset (or of one underlying):

$$\left\{ \begin{array}{ll} -\alpha \leq P_0(t)S_0(t) & \text{where } \alpha \geq 0 \text{ is the "Maximum Borrowing Amount"} \\ \beta W(t) \leq P(t)S(t) & \text{where } 0 \leq \beta \geq 1 \text{ is the "Minimum Target Allocation"} \end{array} \right. \tag{4.14}$$

These contraints are summarized by :

$$\forall\, t \geq 0,\ \ \beta W(t) \ \leq\ P(t)S(t) \ \leq\ W(t) + \alpha$$

We thus set $U(t,S,W) := [\beta W, W+\alpha]$ and $V(t,S) := [-1,+1]$.

More generally, we consider maps of the form

$$U(t,S,W) := [u^\flat(t,S,W), u^\sharp(t,S,W)]$$

and we associate with any feedback $\widetilde{u}(t,S,W)$ the truncated feedback

$$\widetilde{u}^\natural(t,S,W) \ :=\ \min\Big(\max\big(u^\flat(t,S,W), \widetilde{u}(t,S,W)\big), u^\sharp(t,S,W)\Big)$$

which is a selection of the set-valued map U :

$$\forall\, t,S,W,\ \ \widetilde{u}^\natural(t,S,W) \ \in\ [u^\flat(t,S,W), u^\sharp(t,S,W)]$$

More precisely

- If $\widetilde{u}(t,S,W) \leq u^\flat(t,S,W)$, then $\widetilde{u}^\natural(t,S,W) \ :=\ u^\flat(t,S,W)$,
- If $u^\flat(t,S,W) \leq \widetilde{u}(t,S,W) \leq u^\sharp(t,S,W)$, then $\widetilde{u}^\natural(t,S,W) \ :=\ \widetilde{u}(t,S,W)$,
- If $\widetilde{u}(t,S,W) \geq u^\sharp(t,S,W)$, then $\widetilde{u}^\natural(t,S,W) \ :=\ u^\sharp(t,S,W)$.

We introduce a *floor function* $\mathbf{b}(t,S) \geq 0$ and the *cushion* $W(t) - \mathbf{b}(t,S)$, subjected to the constraint

$$\forall\, t \geq 0,\ \ W(t) - \mathbf{b}(t,S) \ \geq\ 0$$

Example : $\mathbf{b}(t,S) := \kappa e^{-\rho^\sharp(T-t)}$ (does not depend of S).

Example: If $U(t, S, W) := [\beta W, W + \alpha]$ and if the feedback is the standard Merton Constant Proportion Portfolio Insurance (CPPI) feedback

$$\widetilde{u}_M(t, W) \ := \ m(W - \mathbf{b}(t))$$

then the truncated Merton feedback is

$$\widetilde{u}_M^\natural(t, W) \ := \ \min\left(\max\left(\beta W, m(W - \mathbf{b}(t))\right), W + \alpha\right) \ \blacksquare$$

The evolution $t \mapsto (S(t), W(t))$ is governed by the controlled tychastic system

$$\left\{\begin{array}{lll} (i) & S'(t) \ = \ \mu(t)S(t) + \sigma(t)S(t)v(t) \\ (ii) & W'(t) \ = \ r_0 W(t) + u(t)(\mu(t) - r_0) + \sigma(t)u(t)v(t) \\ & u(t) \ \in \ [u^\flat(t, S, W), u^\sharp(t, S, W)] \ \& \ v(t) \ \in \ [-1, +1] \end{array}\right. \tag{4.15}$$

controlled by the exposure $u(t) := P(t)S(t)$ of the portfolio.

We set $\mathbf{c}(0, S) := \mathbf{b}(T, S)$ and $\mathbf{c}(t, S) := +\infty$ for all $0 \leq t < T$.

We consider the Guaranteed Capture Bassin of the epigraph of the function $\mathbf{c}$ viable in the epigraph of $\mathbf{b}$ under the controlled tychastic system (4.15), p.135.

Theorem 4.4.1. [***Partial Differential Equation of the Viabilist Portfolio Insurance under Financial Constraints***] *The Guaranteed Capture Bassin of the epigraph of the function* $\mathbf{c}$ *viable in the epigraph of* $\mathbf{b}$ *under the controlled tychastic system (4.15), p.135 is the epigraph of the function* $(t, S) \mapsto W(t, S)$*, which is the unique solution (in an adequate generalized sense) of a free boundary problem for the following (nonlinear) partial differential equation.*
Let us set

$$\widetilde{u}_\Delta^\natural(t, S, W) \ = \left\{\begin{array}{l} u^\flat(t, S, W) \text{ if } \mu(t) - r_0 < -\sigma(t) \\ \min\left(\max\left(u^\flat(t, S, W), S\frac{\partial W(t,S)}{\partial S}\right), u^\sharp(t, S, W)\right) \\ \text{if } \mu(t) - r_0 \in [-\sigma(t), +\sigma(t)] \\ u^\sharp(t, S, W) \text{ if } \sigma(t) \ < \ \mu(t) - r_0 \end{array}\right. \tag{4.16}$$

Then W *is the smallest solution larger than the floor* $\mathbf{b}$ *of*

$$\frac{\partial W}{\partial t} + \frac{\partial W}{\partial S}\mu(t)S + \sigma(t)\left|S\frac{\partial W}{\partial S} - \widetilde{u}_\Delta^\natural(t, S, W)\right| - (\mu(t) - r_0)\widetilde{u}_\Delta^\natural(t, S, W) \ \leq \ r_0 W \tag{4.17}$$

satisfying the final condition $W(T, S) = \mathbf{b}(T, S)$.

The Δ-rule stating that the amount of shares is given by the partial derivative $\dfrac{\partial W(t,S)}{\partial S}$ boils down to taking for feedback

$$\widetilde{u}_{\Delta}(t,S,W) \ := \ S\frac{\partial W(t,S)}{\partial S} \ \text{ implies that } \ \widetilde{P}_{\Delta}(t,S,W) \ := \ \frac{\partial W(t,S)}{\partial S}$$

that we did truncate.

Observe that partial differential equation (4.17), p.135 has discontinuous coefficients and splits in three cases:

1. $\mu(t)-r_0 \in [-\sigma(t),+\sigma(t)]$ and $S\dfrac{\partial W(t,S)}{\partial S} \in [u^{\flat}(t,S,W),u^{\sharp}(t,S,W)]$. Then (4.17) boils down to

$$\frac{\partial W}{\partial t} + r_0\frac{\partial W}{\partial S}S \ \leq \ r_0 W$$

2. $\mu(t) - r_0 < -\sigma(t)$ or $-\sigma(t) \leq \mu(t) - r_0 < \sigma(t)$ and $S\dfrac{\partial W(t,S)}{\partial S} < u^{\flat}(t,S,W)$. Then (4.17) boils down to

$$\frac{\partial W}{\partial t} + \frac{\partial W}{\partial S}\mu(t)S + \sigma(t)\left|S\frac{\partial W}{\partial S} - u^{\flat}(t,S,W)\right| - (\mu(t)-r_0)u^{\flat}(t,S,W) \ \leq \ r_0 W$$

3. $\sigma(t) \ < \ \mu(t) - r_0$ or $-\sigma(t) \leq \mu(t) - r_0 < \sigma(t)$ and $S\dfrac{\partial W(t,S)}{\partial S} > u^{\sharp}(t,S,W)$.

   Then (4.17) boils down to

$$\frac{\partial W}{\partial t} + \frac{\partial W}{\partial S}\mu(t)S + \sigma(t)\left|S\frac{\partial W}{\partial S} - u^{\sharp}(t,S,W)\right| - (\mu(t)-r_0)u^{\sharp}(t,S,W) \ \leq \ r_0 W$$

Proof (Sketch of the) — The tangential condition characterizing the guaranteed capture basin of $\mathcal{E}p(\mathbf{c})$ viable in $\mathcal{E}p(\mathbf{b})$ under the controlled tychastic system (4.15), p.135 states that for any $(t,S,W) \in \mathcal{E}p(W)$, for any $v \in [-1,+1]$, there exists $u \in U(t,S,W)$ such that

$$(1, S + \sigma S v, r_0 W + u(\mu - r_0) + \sigma u v) \ \in \ T_{\mathcal{E}p(W)}(t,S,W)$$

By definition,

$$T_{\mathcal{E}p(W)}(t, S, W(t, S) \ = \ \mathcal{E}p(D_{\uparrow} W(t, S))$$

Therefore, this tangential condition can be rewritten

$$\inf_{u \in U(t,S,W)} \sup_{v \in [-1+1]} (D_{\uparrow} W(t, S)(1, S + \sigma S v) - u(\mu - r_0) - \sigma u v)) \ \leq \ r_0 W$$

For simplicity, we assume that the solution is differentiable. In this case,

$$(\tau, v) \mapsto D_{\uparrow} W(t, S)(v) = \frac{\partial W}{\partial t} \tau + \frac{\partial W}{\partial S} v$$

Hence

$$\frac{\partial W}{\partial t} + \frac{\partial W}{\partial S} \mu(t) S + \inf_{u \in U(t,S,W)} \left(\sup_{v \in [-1+1]} \sigma(t) \left(S \frac{\partial W}{\partial S} - u \right) v - (\mu(t) - r_0) u \right) \ \leq \ r_0 W$$

which can be rewritten in the form

$$\frac{\partial W}{\partial t} + \frac{\partial W}{\partial S} \mu(t) S + \inf_{u \in U(t,S,W)} \left(\sigma(t) \left| S \frac{\partial W}{\partial S} - u \right| - (\mu(t) - r_0) u \right) \ \leq \ r_0 W$$

The convex function

$$u \mapsto \sigma(t) \left| S \frac{\partial W}{\partial S} - u \right| - (\mu(t) - r_0) u$$

is strictly increasing if $\mu(t) - r_0 < -\sigma(t)$, strictly decreasing if $\sigma(t) \ < \ \mu(t) - r_0$ and, if $\mu(t) - r_0 \in [-\sigma(t), +\sigma(t)]$, is decreasing before $S\frac{\partial W}{\partial S}$ and increasing after. In this case, it achieves its minimum at $S\frac{\partial W}{\partial S}$.

Therefore, in the first case, it achieves its minimum at the lower bound $u^{\flat}(t, s, W)$, in the second case at the upper bound $u^{\sharp}(t, s, W)$ and in the third case, at $S\frac{\partial W}{\partial S}$ if this values lies inside $U(t, S, W)$ and otherwise, at the appropriate bound of this interval. This minimum is thus described by formula (4.16), p.135. ■

Theorem 4.4.2. [***Partial Differential Equation of the Merton CPPI under Financial Constraints***] *Let us set*

$$\widetilde{u}_M^\natural(t,W) \ := \ \min\left(\max\left(\beta W, m(W-\mathbf{b}(t))\right), W+\alpha\right)$$

The Guaranteed Capture Bassin of the epigraph of the function $\mathbf{c}$ *(viable in* $\mathbb{R}_+^2$ *under the controlled tychastic system (4.15), p.135 is the epigraph of the function* $(t,S)\mapsto W(t,S)$*, which is the unique solution (in an adequate generalized sense) of a free boundary problem for the following (nonlinear) partial differential equation.*
Then W *is the smallest on negative solution of*

$$\frac{\partial W}{\partial t}+\frac{\partial W}{\partial S}\mu(t)S+\sigma(t)\left|S\frac{\partial W}{\partial S}-\widetilde{u}_M^\natural(t,W)\right|-(\mu(t)-r_0)\widetilde{u}_M^\natural(t,W) \ \leq \ r_0 W \qquad (4.18)$$

satisfying the final condition $W(T,S)=\mathbf{b}(T,S)$.

This equation splits in three cases:

1. $m(W-\mathbf{b}(t))\in[\beta W, W+\alpha]$. Then (4.17) boils down to

$$\frac{\partial W}{\partial t}+\frac{\partial W}{\partial S}\mu(t)S+\sigma(t)\left|S\frac{\partial W}{\partial S}-m(W-\mathbf{b}(t))\right|-m(W-\mathbf{b}(t))(\mu(t)-r_0) \ \leq \ r_0 W$$

2. $m(W-\mathbf{b}(t))<\beta W$. Then (4.17) boils down to

$$\frac{\partial W}{\partial t}+\frac{\partial W}{\partial S}\mu(t)S+\sigma(t)\left|S\frac{\partial W}{\partial S}-\beta W\right|-(\mu(t)-r_0)\beta W \ \leq \ r_0 W$$

3. $m(W-\mathbf{b}(t))>W+\alpha$.
   Then (4.17) boils down to

$$\frac{\partial W}{\partial t}+\frac{\partial W}{\partial S}\mu(t)S+\sigma(t)\left|S\frac{\partial W}{\partial S}-W-\alpha\right|-(\mu(t)-r_0)(W+\alpha) \ \leq \ r_0 W$$

Example

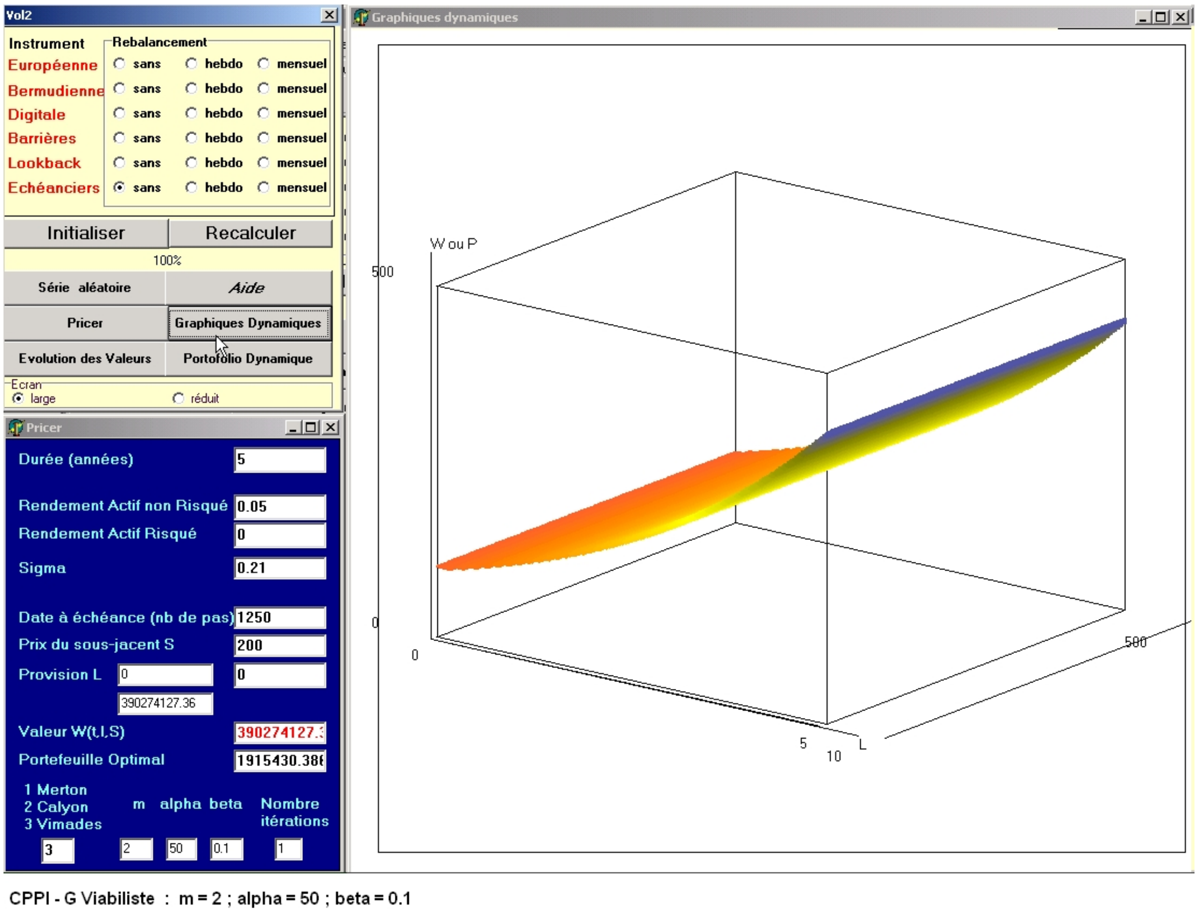
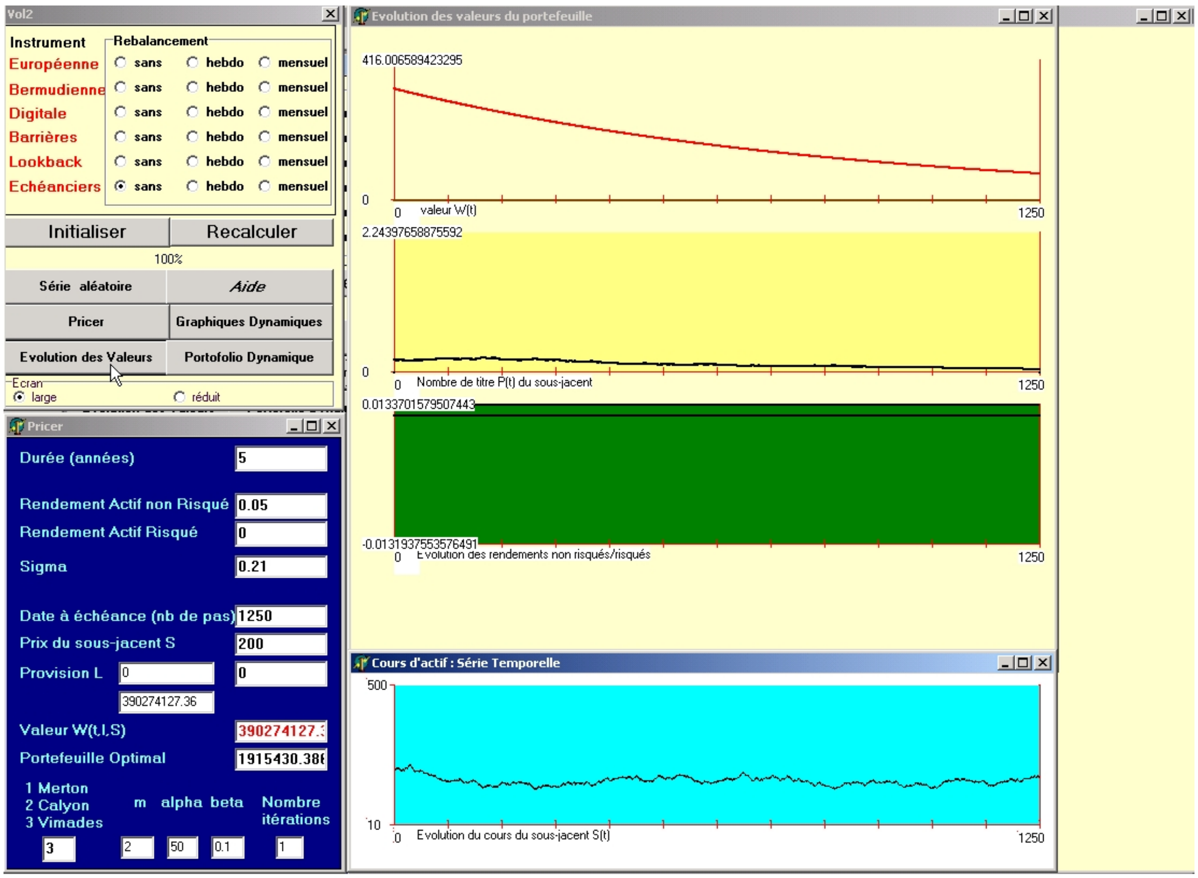


Figure 4.4.3. The Value Function $W(t,S)$. Case when $\alpha = 50,\ \beta = 0.1$.



CPPI - G Viabiliste : m = 2 ; alpha = 50 ; beta = 0.1

Figure 4.4.4. Case when $\alpha = 50,\ \beta = 0.1$.

Chapter 5

The Main Viability and Invariance Theorems

5.1 Bilateral Fixed Point Characterization of Kernels and Basins

5.1.1 Bilateral Fixed Point Characterization of Viability Kernels

We shall start our presentation of kernels and basins' properties by a simple and important algebraic property:

Theorem 5.1.1. [***The Fundamental Characterization of Viability Kernels***] *Let $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0,\infty;X)$ be an evolutionary system, $K \subset X$ be a environment and $C \subset K$ be a target.* ***The viability kernel*** $\mathrm{Viab}_{\mathcal{S}}(K,C)$ ***of*** K ***outside the target*** C ***is the unique subset between*** C ***and*** K ***that is both***

1. ***viable outside*** C *(and is the largest subset $D \subset K$ viable outside C),*
2. ***isolated in*** K *(and is the smallest subset $D \supset C$ isolated in K):*

$$\mathrm{Viab}_{\mathcal{S}}(K, \mathrm{Viab}_{\mathcal{S}}(K,C)) = \mathrm{Viab}_{\mathcal{S}}(K,C) = \mathrm{Viab}_{\mathcal{S}}(\mathrm{Viab}_{\mathcal{S}}(K,C),C) \tag{5.1}$$

The viability kernel satisfies the properties of both the subsets viable outside a target and of isolated subsets in a environment, and is the unique one to do so.

This statement is at the root of uniqueness properties of solutions to some Hamilton-Jacobi-Bellman partial differential equations whenever the epigraph of a solution is a viability kernel of the epigraph of a function outside the epigraph of another function.

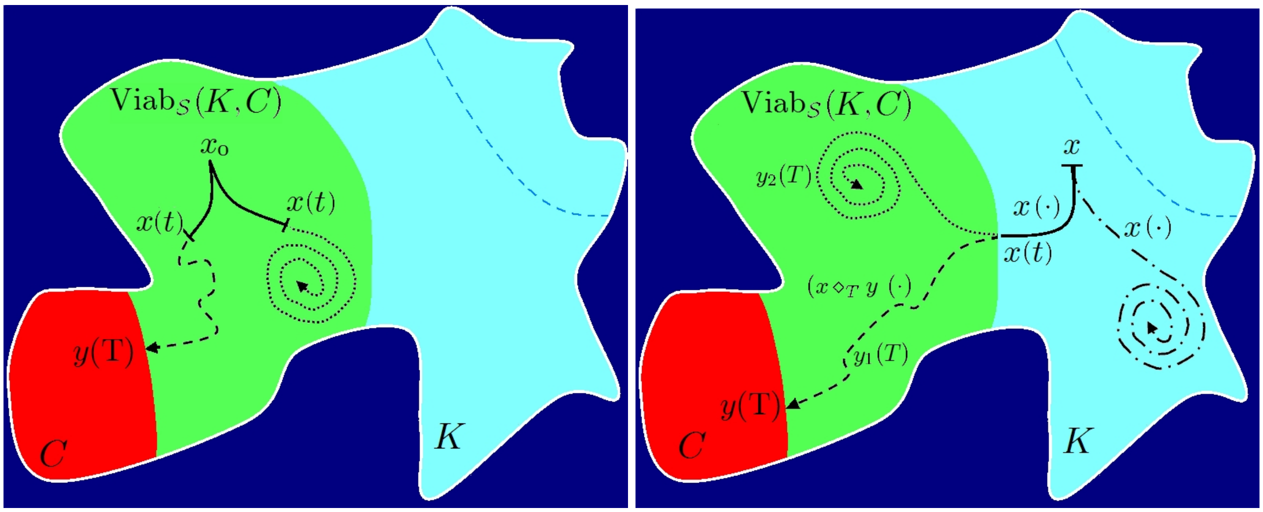


Figure 5.1: [Illustration of the proof of Theorem 5.1.1]

Proof of Theorem 5.1.1 — We begin by proving the two following statements:

1. The translation property implies that the viability kernel $\mathrm{Viab}_{\mathcal{S}}(K,C)$ is viable outside C:

$$\mathrm{Viab}_{\mathcal{S}}(K,C) \ \subset \ \mathrm{Viab}_{\mathcal{S}}(\mathrm{Viab}_{\mathcal{S}}(K,C),C)$$

   Take $x_0 \in \mathrm{Viab}_{\mathcal{S}}(K,C)$ and prove that there exists an evolution $x(\cdot) \in \mathcal{S}(x_0)$ starting at x_0 viable in $\mathrm{Viab}_{\mathcal{S}}(K,C)$ until it possibly reaches C. Indeed, there exists an evolution $x(\cdot) \in \mathcal{S}(x_0)$ viable in K until some time $T \geq 0$ either finite when it reaches C or infinite. Then for all $t \in [0,T[$, the translation $y(\cdot) := \kappa(-t)x(\cdot)$ of $x(\cdot)$ defined by $y(\tau) := x(t+\tau)$ is an evolution $y(\cdot) \in \mathcal{S}(x(t))$ starting at $x(t)$ and viable in K until it reaches C at time $T-t$. Hence $x(t)$ does belong to $\mathrm{Viab}_{\mathcal{S}}(K,C)$ for every $t \in [0,T[$.

2. The concatenation property implies that the viability kernel $\mathrm{Viab}_{\mathcal{S}}(K,C)$ is isolated in K:

$$\mathrm{Viab}_{\mathcal{S}}(K,\mathrm{Viab}_{\mathcal{S}}(K,C)) \ \subset \ \mathrm{Viab}_{\mathcal{S}}(K,C)$$

   Let x belong to $\mathrm{Viab}_{\mathcal{S}}(K,\mathrm{Viab}_{\mathcal{S}}(K,C))$. There exists at least one evolution $x(\cdot) \in \mathcal{S}(x)$ that would either remain in K or reach the viability kernel $\mathrm{Viab}_{\mathcal{S}}(K,C)$ in finite time. In this case, it can be concatenated with an evolution either remaining in $\mathrm{Viab}_{\mathcal{S}}(K,C) \subset K$ or reaching the target C in finite time. This implies that $x \in \mathrm{Viab}_{\mathcal{S}}(K,C)$.

We now observe that the map $(K,C) \mapsto \mathrm{Viab}_{\mathcal{S}}(K,C)$ satisfies

$$\begin{cases} (i) & C \subset \mathrm{Viab}_{\mathcal{S}}(K,C) \subset K \\ (ii) & (K,C) \mapsto \mathrm{Viab}_{\mathcal{S}}(K,C) \text{ is increasing} \end{cases} \tag{5.2}$$

in the sense that if $K_1 \subset K_2$ and $C_1 \subset C_2$, then $\mathrm{Viab}_{\mathcal{S}}(K_1,C_1) \subset \mathrm{Viab}_{\mathcal{S}}(K_2,C_2)$.

Setting $\mathcal{A}(K,C) := \mathrm{Viab}_{\mathcal{S}}(K,C)$, the other statements follow from general algebraic Lemma 5.1.2 below. ■

Lemma 5.1.2. [***Uniqueness of Bilateral Fixed Points***] *Let us consider a map* $\mathcal{A} : (K,C) \mapsto \mathcal{A}(K,C)$ *satisfying*

$$\begin{cases} (i) & C \subset \mathcal{A}(K,C) \subset K \\ (ii) & (K,C) \mapsto \mathcal{A}(K,C) \text{ is increasing} \end{cases} \tag{5.3}$$

1. *If* $\mathcal{A}(K,C) = \mathcal{A}(\mathcal{A}(K,C),C)$*, it is the largest fixed point of the map* $D \mapsto \mathcal{A}(D,C)$ *between* C *and* K*,*
2. *If* $\mathcal{A}(K,C) = \mathcal{A}(K,\mathcal{A}(K,C))$*, it is the smallest fixed point of the map* $E \mapsto \mathcal{A}(K,E)$ *between* C *and* K*.*

Then, any subset D *between* C *and* K *satisfying*

$$D = \mathcal{A}(D,C) \text{ and } \mathcal{A}(K,D) = D$$

is the unique bilateral fixed point D *between* C *and* K *of the map* $\mathcal{A}$ *in the sense that:*

$$\mathcal{A}(K,D) = D = \mathcal{A}(D,C)$$

and is equal to $\mathcal{A}(K,C)$*.*

Proof of Lemma 5.1.2 — If $D = \mathcal{A}(D,C)$ is a fixed point of $D \mapsto \mathcal{A}(D,C)$, we then deduce that $\mathcal{A}(D,C) \subset \mathcal{A}(K,C)$, so that *whenever* $\mathcal{A}(K,C) = \mathcal{A}(\mathcal{A}(K,C),C)$*, we deduce that* $\mathcal{A}(K,C)$ *is the largest fixed point of* $D \mapsto \mathcal{A}(D,C)$ contained in K. In the same way, *if* $\mathcal{A}(K,\mathcal{A}(K,C)) = \mathcal{A}(K,C)$*, then* $\mathcal{A}(K,C)$ *is the smallest fixed points of* $E \mapsto \mathcal{A}(K,E)$ containing C. Furthermore, equalities

$$\mathcal{A}(K,D) \ = \ D \ = \ \mathcal{A}(D,C)$$

imply that $D = \mathcal{A}(K,C)$ because the monotonicity property implies that

$$\mathcal{A}(K,C) \subset \mathcal{A}(K,D) \subset D \subset \mathcal{A}(D,C) \subset \mathcal{A}(K,C) \ \blacksquare$$

5.1.2 Bilateral Fixed Point Characterization of Invariance Kernels

This existence and uniqueness of a "bilateral fixed point" is shared by the invariance kernel outside a target, the capture basin and the absorption basin of a target that satisfy property (5.3), and thus, the conclusions of Lemma 5.1.2:

Theorem 5.1.3. [***Characterization of Kernels and Basins as Unique Bilateral Fixed Point***] *Let* $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0,\infty;X)$ *be an evolutionary system,* $K \subset X$ *be a environment and* $C \subset K$ *be a target.*

1. *The viability kernel* $\mathrm{Viab}_{\mathcal{S}}(K,C)$ *of a subset* K *with target* $C \subset K$ *is the* unique *bilateral fixed point* D *between* C *and* K *of the map* $(K,C) \mapsto \mathrm{Viab}_{\mathcal{S}}(K,C)$ *in the sense that*
$$D = \mathrm{Viab}_{\mathcal{S}}(K,D) = \mathrm{Viab}_{\mathcal{S}}(D,C)$$
2. *The invariance kernel* $\mathrm{Inv}_{\mathcal{S}}(K,C)$ *of a subset* K *with target* $C \subset K$ *is the* unique *bilateral fixed point* D *between* C *and* K *of the map* $(K,C) \mapsto \mathrm{Inv}_{\mathcal{S}}(K,C)$ *in the sense that*
$$D = \mathrm{Inv}_{\mathcal{S}}(K,D) = \mathrm{Inv}_{\mathcal{S}}(D,C)$$

The same properties are shared by the maps $(K,C) \mapsto \mathrm{Capt}_{\mathcal{S}}(K,C)$ *and* $(K,C) \mapsto \mathrm{Abs}_{\mathcal{S}}(K,C)$ *.*

The consequences of these simple observations are important:

Lemma 5.1.4. [***Union of Targets and Intersection of Environments***]

$$\left\{\begin{array}{ll} (i) & \mathrm{Viab}_{\mathcal{S}}\left(K, \bigcup_{i\in I} C_i\right) = \bigcup_{i\in I} \mathrm{Viab}_{\mathcal{S}}(K,C_i) \\ (ii) & \mathrm{Inv}_{\mathcal{S}}\left(\bigcap_{i\in I} K_i, C\right) = \bigcap_{i\in I} \mathrm{Inv}_{\mathcal{S}}(K_i,C) \end{array}\right. \tag{5.4}$$

The bilateral fixed point properties imply the stability under union and intersections of viability kernels:

Lemma 5.1.5. [***Union of Viable Sets & Intersection of Isolated Sets***] *The union of subsets $K_i \supseteq C$ viable outside C is viable outside a target C is viable outside C and the intersection of subsets $C_i \subset K$ isolated in K is isolated in K. More generally, the following equalities hold true:*

$$\left\{\begin{array}{ll} (i) & \bigcup_{i\in I}\mathrm{Viab}_{\mathcal{S}}(K_i, C) = \mathrm{Viab}_{\mathcal{S}}\left(\bigcup_{i\in I}\mathrm{Viab}_{\mathcal{S}}(K_i, C), C\right) \\ & \\ (ii) & \bigcap_{i\in I}\mathrm{Viab}_{\mathcal{S}}(K, C_i) = \mathrm{Viab}_{\mathcal{S}}\left(K, \bigcap_{i\in I}\mathrm{Viab}_{\mathcal{S}}(K, C_i))\right) \end{array}\right. \tag{5.5}$$

The same results are valid for the invariance kernels and the capture and absorption basins.

Proof of Lemma 5.1.5 — Indeed, the first equality follows from

$$\mathrm{Viab}_{\mathcal{S}}(K_i, C) = \mathrm{Viab}_{\mathcal{S}}(\mathrm{Viab}_{\mathcal{S}}(K_i, C), C) \ \subset \ \mathrm{Viab}_{\mathcal{S}}\left(\bigcup_{i\in I}\mathrm{Viab}_{\mathcal{S}}(K_i, C), C\right)$$

and the second one from

$$\mathrm{Viab}_{\mathcal{S}}(K, C_i) = \mathrm{Viab}_{\mathcal{S}}(K, \mathrm{Viab}_{\mathcal{S}}(K, C_i)) \ \supset \ \mathrm{Viab}_{\mathcal{S}}\left(K, \bigcap_{i\in I}\mathrm{Viab}_{\mathcal{S}}(K, C_i)\right) \ \blacksquare$$

Lemma 5.1.6. [***Viable or Invariant Boundaries***] *If the boundary of a closed subset is viable (respectively invariant), so is the subset itself.*

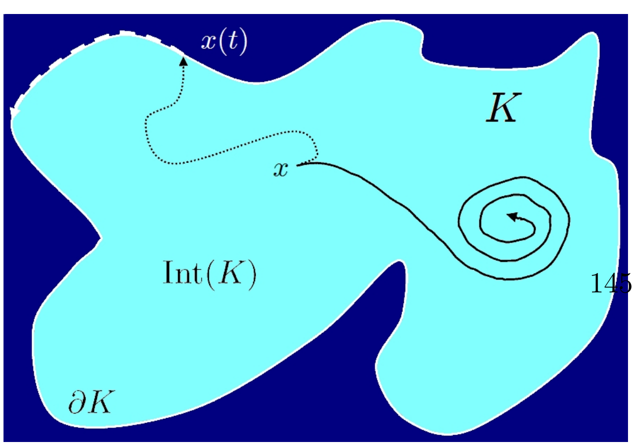


Figure 5.2: [**Proof of Lemma 5.1.6**]

Indeed, if $x \in \mathrm{Int}(K)$, any evolution $x(\cdot) \in \mathcal{S}(x)$ starting from x is either viable in the interior of K, and thus in K, or else, leaves K at a finite time T at a point $x(T)$ of its boundary. Then we can concatenate the evolution with an evolution viable in ∂K, so that the concatenated evolution is viable in K. Then K is viable. $\blacksquare$

5.2 Continuity Properties of Evolutionary Systems

In order to go further in the characterization of viability and invariance kernels with targets in terms of properties easier to check, we need to bring in the forefront some continuity requirements on the evolutionary system $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0, \infty; X)$. First, both the state space X and the evolutionary $\mathcal{C}(0, \infty; X)$ have to be complete topological spaces.

Lemma 5.2.1. [***The Evolutionary Space***] *Assume that the state space X is a complete metric space. We supply the space $\mathcal{C}(0, \infty; X)$ of continuous evolutions with the "compact topology": A sequence of continuous evolutions $x_n(\cdot) \in \mathcal{C}(0, \infty; X)$ converges to the continuous evolution $x(\cdot)$ as $n \to +\infty$ if for every $T > 0$, the sequence $\sup_{t\in[0,T]} d(x_n(t), x(t))$ converges to 0. It is a complete metrizable space. The Ascoli Theorem states that a subset $\mathcal{H}$ is compact if and only if it is closed, equicontinuous and for any $t \in \mathbb{R}_+$, the subset $\mathcal{H}(t) := \{x(t)\}_{x(\cdot)\in\mathcal{H}}$ is compact in X.*

Stability, a polysemous word, means that the solution of a problem depends continuously upon its data. Here, for evolutionary systems, the data are principally the initial states: In this case, stability means that the set of solutions depends "continuously" on the initial state. We recall that a deterministic system $\mathcal{S} : X \mapsto \mathcal{C}(0, \infty; X)$ is continuous at some $x \in X$ if it maps any sequence $x_n \in X$ converging to x to a sequence $\mathcal{S}(x_n)$ converging to $\mathcal{S}(x)$. However, when the evolutionary system $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0, \infty; X)$ is no longer single-valued, there are several ways of describing the convergence of the set $\mathcal{S}(x_n)$ to the set $\mathcal{S}(x)$.

We shall use in these lecture notes only two of them, that we present in the context of evolutionary systems (Set-Valued Analysis investigates a lot more...). We begin with the notion of upper semicompactness:

Definition 5.2.2. [***Upper Semicompactness***] *Let $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0, \infty; X)$ be an evolutionary system, where both the state space X and the evolutionary space $\mathcal{C}(0, \infty; X)$ are topological spaces. The evolutionary system is said to be* upper semicompact *at x if for every sequence $x_n \in X$ converging to x and for every sequence $x_n(\cdot) \in \mathcal{S}(x_n)$, there exists a subsequence $x_{n_p}(\cdot)$ converging to some $x(\cdot) \in \mathcal{S}(x)$. It is said to be* upper semicompact *if it is upper semicompact at every point $x \in X$ where $\mathcal{S}(x)$ is not empty.*

Before using this property, we need to provide examples of evolutionary system enjoying it: This is the case for Marchaud differential inclusions:

Definition 5.2.3. [***Marchaud Set-Valued Maps***] *We say that F is a* Marchaud map *if*

$$\left\{ \begin{array}{ll} (i) & \textit{the graph and the domain of } F \textit{ are nonempty and closed} \\ (ii) & \textit{the values } F(x) \textit{ of } F \textit{ are convex} \\ (iii) & \exists\, c > 0 \text{ such that } \forall x \in X,\ \|F(x)\| := \sup_{v \in F(x)} \|v\| \leq c(\|x\| + 1) \end{array} \right. \quad (5.6)$$

Marchaud was with Zaremba among the first to study what will become known under the name of differential inclusions:

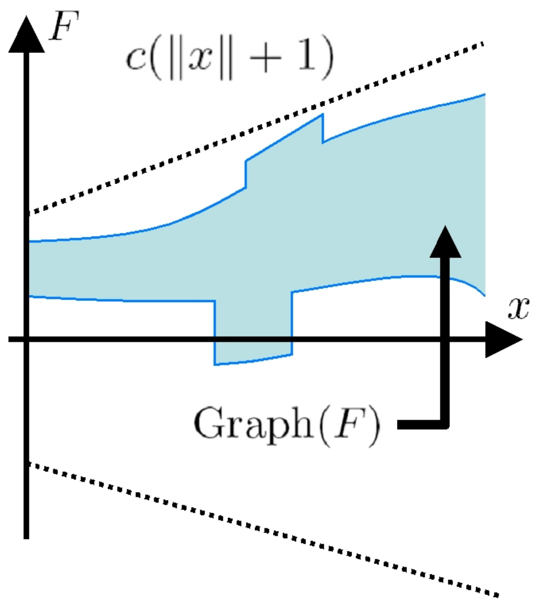


Figure 5.3: **[Marchaud map]**

The (difficult) Stability Theorem states that the set of solutions depends continuously upon the initial states in the upper semicompact sense:

Theorem 5.2.4. [***Upper Semicompactness of Marchaud Evolutionary Systems***] *If $F : X \rightsquigarrow X$ is Marchaud, the solution map $\mathcal{S}$ is an upper semicompact evolutionary system.*

The other way to take into account the idea of continuity in the case of evolutionary systems is by introducing the following concept:

Definition 5.2.5. [***Lower Semicontinuity***] *Let $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0, \infty; X)$ be an evolutionary system, where both the state space X and the evolutionary space $\mathcal{C}(0, \infty; X)$ are topological spaces. The evolutionary system is said to be* lower semicontinuous *at x if for every sequence $x_n \in X$ converging to x and for every sequence $x(\cdot) \in \mathcal{S}(x)$ (thus assumed to be nonempty), there exists a sequence $x_n(\cdot) \in \mathcal{S}(x_n)$ converging to $x(\cdot) \in \mathcal{S}(x)$. It is said to be* lower semicontinuous *if it is lower semicontinuous at every point $x \in X$ where $\mathcal{S}(x)$ is not empty.*

Warning: An evolutionary system can be upper semicompact at x without being lower semicontinuous and lower semicontinuous at x without being upper semicompact. If the evolutionary system is deterministic, lower semicontinuity coincides with continuity and upper semicompactness coincides with "properness" of single-valued maps (in the sense of Bourbaki).

Recall that a single-valued map $\mathbf{f} : X \mapsto Y$ is said to be λ-*Lipschitz* if for any $x_1,\ x_2 \in X$, $d(\mathbf{f}(x_1), \mathbf{f}(x_2)) \leq \lambda d(x_1, x_2)$. In the case of normed vector spaces, denoting by B the unit ball of the vector space, this inequality can be translated in the form $\mathbf{f}(x_1) \in \mathbf{f}(x_2) + \lambda \|x_1 - x_2\| B$

The evolutionary system associated with a Lipschitz differential inclusion is lower semicontinuous:

Definition 5.2.6. [***Lipschitz Maps***] *A set-valued map $F : X \rightsquigarrow Y$ is said to be λ-*Lipschitz*(or Lipschitz for the constant $\lambda > 0$) if*

$$\forall x_1,\ x_2,\ F(x_1)\ \subset\ F(x_2) + \lambda \|x_1 - x_2\| B$$

The evolutionary system $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0, \infty; X)$ associated with a Lipschitz set-valued map is called a Lipschitz evolutionary system.

The Filippov Theorem implies that Lipschitz systems are lower semicontinuous:

Theorem 5.2.7. [***Lower Semicontinuity of Lipschitz Evolutionary Systems***] *If $F : X \rightsquigarrow X$ is Lipschitz, the associated evolutionary system $\mathcal{S}$ is lower semicontinuous.*

Under appropriate topological assumptions, we can prove that inverse images and cores of closed subsets of evolutions are closed.

Definition 5.2.8. [***Closedness of Inverse Images***] *Let $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0,\infty;X)$ be an upper semicompact evolutionary system. Then for any subset $\mathcal{H} \subset \mathcal{C}(0,\infty;X)$,*

$$\overline{\mathcal{S}^{-1}(\mathcal{H})} \subset \mathcal{S}^{-1}(\overline{\mathcal{H}})$$

Consequently, the inverse images $\mathcal{S}^{-1}(\mathcal{H})$ under $\mathcal{S}$ of any closed subset $\mathcal{H} \subset \mathcal{C}(0,\infty;X)$ is closed.

Furthermore, the evolutionary system $\mathcal{S}$ maps compact sets $K \subset X$ to compact sets $\mathcal{H} \subset \mathcal{C}(0,\infty;X)$.

Proof — Let us consider a subset $\mathcal{H} \subset \mathcal{C}(0,\infty;X)$, a sequence of elements $x_n \in \mathcal{S}^{-1}(\mathcal{H})$ converging to some x and prove that x belongs to $\mathcal{S}^{-1}(\mathcal{H})$. Hence there exist elements $x_n(\cdot) \in \mathcal{S}(x_n) \cap \mathcal{H}$. Since $\mathcal{S}$ is upper semicompact, there exists a subsequence $x_{n'}(\cdot) \in \mathcal{S}(x_{n'})$ converging to some $x(\cdot) \in \mathcal{S}(x)$. It belongs also to the closure of $\mathcal{H}$, so that $x \in \mathcal{S}^{-1}(\overline{\mathcal{H}})$.

Take now any compact subset $K \subset X$. For proving that $\mathcal{S}(K)$ is compact, take any sequence $x_n(\cdot) \in \mathcal{S}(x_n)$ where $x_n \in K$. Since K is compact, a subsequence $x_{n'}$ converges to some $x \in K$ and since $\mathcal{S}$ is upper semicompact, a subsequence $x_{n''}(\cdot) \in \mathcal{S}(x_{n''})$ converges to some $x(\cdot) \in \mathcal{S}(x)$. ■

For cores, we obtain

Theorem 5.2.9. [***Closedness of Cores***] *Let $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0,\infty;X)$ be a lower semicontinuous evolutionary system. Then for any subset $\mathcal{H} \subset \mathcal{C}(0,\infty;X)$,*

$$\overline{\mathcal{S}^{\ominus 1}(\mathcal{H})} \subset \mathcal{S}^{\ominus 1}(\overline{\mathcal{H}})$$

Consequently, the core $\mathcal{S}^{\ominus 1}(\mathcal{H})$ under $\mathcal{S}$ of any closed subset $\mathcal{H} \subset \mathcal{C}(0,\infty;X)$ is closed.

Proof — Let us consider a closed subset $\mathcal{H} \subset \mathcal{C}(0,\infty;X)$, a sequence of elements $x_n \in \mathcal{S}^{\ominus 1}(\mathcal{H})$ converging to some x and prove that x belongs to $\mathcal{S}^{\ominus 1}(\mathcal{H})$. We have to prove that any $x(\cdot) \in \mathcal{S}(x)$ belongs to $\mathcal{H}$. But since $\mathcal{S}$ is lower semicontinuous, there exists a sequence of elements $x_n(\cdot) \in \mathcal{S}(x_n) \subset \mathcal{H}$ converging to $x(\cdot) \in \overline{\mathcal{H}}$. Therefore $\mathcal{S}(x) \subset \overline{\mathcal{H}}$, i.e., $x \in \mathcal{S}^{\ominus 1}(\overline{\mathcal{H}})$. ■

5.3 Topological Properties of Viability Kernels and Capture Basins

Recall that the set $\mathcal{V}(K,C)$ of evolutions *viable in K outside C* is defined by (1.5):

$$\left\{ \begin{array}{l} \mathcal{V}(K,C) := \{x(\cdot) \text{ such that } \forall t \geq 0, \;\; x(t) \in K \\ \text{or } \; \exists\, T \geq 0 \text{ such that } x(T) \in C \;\&\; \forall t \in [0,T], \; x(t) \; \in \; K\} \end{array} \right.$$

Actually, the viability kernel of a closed subset with a closed target under an upper semicompact evolutionary subset is closed:

Theorem 5.3.1. [***Closedness of the Viability Kernel***] *Let $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0,\infty;X)$ be an upper semicompact evolutionary system. Then for any constrained subset $K \subset X$ and any target $C \subset K$,*

$$\overline{\mathrm{Viab}_{\mathcal{S}}(K,C)} \; \subset \; \mathrm{Viab}_{\mathcal{S}}(\overline{K},\overline{C})$$

Consequently, if $C \subset K$ and K are closed, so is the viability kernel $\mathrm{Viab}_{\mathcal{S}}(K,C)$ *of K with target C. Furthermore, if $K \backslash C$ is a repeller, the capture basin* $\mathrm{Capt}_{\mathcal{S}}(K,C)$ *of C viable in K under $\mathcal{S}$ is closed.*

Since the viability kernel $\mathrm{Viab}_{\mathcal{S}}(K,C) := \mathcal{S}^{-1}(\mathcal{V}(K,C))$ is the inverse image of the subset $\mathcal{V}(K,C)$ by Definition 1.11.2, the closedness of the viability kernel follows from Theorem 5.2.8 and Lemma 5.3.2:

Lemma 5.3.2. [***Closedness of the Subset of Viable Evolutions***] *Let us consider a constrained subset $K \subset X$ and a (possibly empty) target $C \subset K$. Then*

$$\overline{\mathcal{V}(K,C)} \; \subset \; \mathcal{V}(\overline{K},\overline{C})$$

and consequently, if C and K are closed, the set $\mathcal{V}(K,C)$ of evolutions that are viable in K forever or until they reach the target C in finite time is closed.

Proof — Let us consider a sequence of evolutions $x_n(\cdot) \in \mathcal{V}(K,C)$ converging to some evolution $x(\cdot)$. We have to prove that $x(\cdot)$ belongs to $\mathcal{V}(\overline{K},\overline{C})$, i.e., that it is viable in $\overline{K}$ forever or until it reaches the target $\overline{C}$ in finite time. Indeed,

1. either for any $T > 0$ and any $N > 0$, there exist $n \geq N$, $t_n \geq T$ and an evolution $x_n(\cdot)$ for which $x_n(t) \in K$ for every $t \in [0, t_n]$,
2. or there exit $T > 0$ and $N > 0$ such that for any $t \geq T$ and $n \geq N$ and any evolution $x_n(\cdot)$, there exists $t_n \leq t$ such that $x_n(t_n) \notin K$.

In the first case, we deduce that for any $T > 0$, $x(T) \in \overline{K}$, so that the limit $x(\cdot)$ is viable in $\overline{K}$ forever.

In the second case, all the solutions $x_n(\cdot)$ leave K before T. This is impossible if evolutions $x_n(\cdot)$ are viable in K forever. Therefore, since $x_n(\cdot) \in \mathcal{V}(K,C)$, they have to reach C before leaving K: There exist $s_n \leq T$ such that

$$x_n(s_n) \in C \;\&\; \forall t \in [0, s_n],\ x_n(t) \ \in\ K$$

Then a subsequence $s_{n'}$ converges to some $S \in [0,T]$. Therefore, for any $s < S$, then $s < s_{n'}$ for n' large enough, so that $x_{n'}(s) \ \in\ K$. By taking the limit, we infer that for every $s < S$, $x(s) \in \overline{K}$. Furthermore, since $x_n(\cdot)$ converges to $x(\cdot)$ uniformly on the compact interval $[0,T]$, then $x_n(s_n)$ converges to $x(S)$, that belongs to $\overline{C}$.

This shows that the limit $x(\cdot)$ belongs to $\mathcal{V}(\overline{K},\overline{C})$. ■

Theorem 5.2.9 implies the closedness of the invariance kernels:

Theorem 5.3.3. [***Closedness of Invariance Kernels***] *Let $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0,\infty;X)$ be a lower semicontinuous evolutionary system. Then for any constrained subset $K \subset X$ and any target $C \subset K$,*

$$\overline{\mathrm{Inv}_{\mathcal{S}}(K,C)} \ \subset\ \mathrm{Inv}_{\mathcal{S}}(\overline{K},\overline{C})$$

Consequently, if $C \subset K$ and K are closed, so is the invariance kernel $\mathrm{Inv}_{\mathcal{S}}(K,C)$ of K with target C.

Therefore, if $K \backslash C$ is a repeller, the absorption basin $\mathrm{Abs}_{\mathcal{S}}(K,C)$ of C invariant in K under $\mathcal{S}$ is closed.

As for interiors of capture and absorption basins, we obtain the following statements:

Theorem 5.3.4. [***Interiors of Capture and Absorption Basins***] *For any constrained subset $K \subset X$ and any target $C \subset K$,*

- *if $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0,\infty;X)$ is lower semicontinuous, then*

$$\mathrm{Capt}_{\mathcal{S}}(\mathrm{Int}(K),\mathrm{Int}(C)) \ \subset \ \mathrm{Int}(\mathrm{Capt}_{\mathcal{S}}(K,C))$$

- *if $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0,\infty;X)$ is upper semicontinuous, then*

$$\mathrm{Abs}_{\mathcal{S}}(\mathrm{Int}(K),\mathrm{Int}(C)) \ \subset \ \mathrm{Int}(\mathrm{Abs}_{\mathcal{S}}(K,C))$$

Consequently, if $C \subset K$ and K are open, so are the capture basin $\mathrm{Capt}_{\mathcal{S}}(K,C)$ and the absorption basin $\mathrm{Capt}_{\mathcal{S}}(K,C)$ whenever the evolutionary system is respectively lower semicontinuous and upper semicompact.

Proof — Observe that, taking the complements, Lemma 3.3.2 implies that if $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0,\infty;X)$ is lower semicontinuous, then Theorem 5.3.3, p.151 implies that

$$\mathrm{Capt}_{\mathcal{S}}(\mathrm{Int}(K),\mathrm{Int}(C)) \ \subset \ \mathrm{Int}(\mathrm{Capt}_{\mathcal{S}}(K,C))$$

since the complement of an invariance kernel is the capture basin of the complements and since the complement of a closure is the interior of the complement, and Theorem 5.3.1, p.150 imply the similar statement for absorption basins. ■

For capture basins, we obtain another closedness property:

Proposition 5.3.5. [***Closedness of Capture Basins***] *If the set-valued map $\overleftarrow{\mathcal{S}}$ is lower semicontinuous and if K is backward invariant, then for any closed subset $C \subset K$,*

$$\mathrm{Capt}_{\mathcal{S}}(\overline{K},\overline{C}) \ \subset \ \overline{\mathrm{Capt}_{\mathcal{S}}(K,C)} \tag{5.7}$$

Proof — Let us take $x \in \mathrm{Capt}_{\mathcal{S}}(\overline{K},\overline{C})$ and an evolution $x(\cdot) \in \mathcal{S}(x)$ viable in $\overline{K}$ until it reaches the target $\overline{C}$ at time $T < +\infty$ at $c := x(T) \in \overline{C}$. Hence the function $t \mapsto y(t) := x(T-t)$ is an evolution $y(\cdot) \in \overleftarrow{\mathcal{S}}(c)$.

Let us consider a sequence of elements $c_n \in C$ converging to c. Since $\overleftarrow{\mathcal{S}}$ is lower semicontinuous, there exist evolutions $y_n(\cdot) \in \overleftarrow{\mathcal{S}}(c_n)$ converging uniformly over compact intervals to $y(\cdot)$. These evolutions $y_n(\cdot)$ are viable in K, since K is assumed to be backward invariant. The evolutions $x_n(\cdot)$ defined by $x_n(t) := y_n(T-t)$ satisfy $x_n(0) = y_n(T) \in K$, $x_n(T) = c_n$ and, for all $t \in [0,T]$, $x_n(t) \in K$. Therefore $x_n(0) := y_n(T)$ belongs to $\mathrm{Capt}(K,C)$ and converges to $x := x(0)$, so that $x \in \overline{\mathrm{Capt}_{\mathcal{S}}(K,C)}$. ■

As a consequence, we obtain the following regularity property of capture basins:

Proposition 5.3.6. [***Topological Regularity of Capture Basins***] *If the set-valued map $\mathcal{S}$ is upper semicompact and the set-valued map $\overleftarrow{\mathcal{S}}$ is lower semicontinuous, if $K = \overline{\mathrm{Int}(K)}$ and $C = \overline{\mathrm{Int}(C)}$, if $K \setminus C$ is a repeller and if* $\mathrm{Int}(K)$ *is backward invariant, then*

$$\mathrm{Capt}_{\mathcal{S}}(K,C) \;=\; \overline{\mathrm{Capt}_{\mathcal{S}}(\mathrm{Int}(K),\mathrm{Int}(C))}; =\; \overline{\mathrm{Int}(\mathrm{Capt}_{\mathcal{S}}(K,C))} \tag{5.8}$$

Proof — Since $K = \overline{\mathrm{Int}(K)}$ and $C = \overline{\mathrm{Int}(C)}$, since $\overleftarrow{\mathcal{S}}$ is lower semicontinuous and since $\mathrm{Int}(K)$ is backward invariant, Proposition 5.3.6, p.153 implies that

$$\mathrm{Capt}_{\mathcal{S}}(K,C) \;=\; \mathrm{Capt}_{\mathcal{S}}(\overline{\mathrm{Int}(K)},\overline{\mathrm{Int}(C)}) \;\subset\; \overline{\mathrm{Capt}_{\mathcal{S}}(\mathrm{Int}(K),\mathrm{Int}(C))}$$

Inclusion

$$\mathrm{Capt}_{\mathcal{S}}(\mathrm{Int}(K),\mathrm{Int}(C)) \;\subset\; \mathrm{Int}(\mathrm{Capt}_{\mathcal{S}}(K,C))$$

follows from Theorem 5.3.4, p.152. On the other hand, since $\mathcal{S}$ is upper semicompact and $K \setminus C$ is a repeller, Theorem 5.3.1, p.150 implies that

$$\overline{\mathrm{Capt}_{\mathcal{S}}(\mathrm{Int}(K),\mathrm{Int}(C))} \;\subset\; \overline{\mathrm{Int}(\mathrm{Capt}_{\mathcal{S}}(K,C))} \;\subset\; \mathrm{Capt}_{\mathcal{S}}(K,C)$$

so that $mbox\mathrm{Capt}_{\mathcal{S}}(K,C) = \overline{\mathrm{Int}(\mathrm{Capt}_{\mathcal{S}}(K,C))}$. ■

Further characterizations require properties of the invariance kernels in terms of closed viable or invariant subsets. For instance:

Proposition 5.3.7. [***Invariance Kernels***] *Let us assume that $C \subset K$ and K are closed,* *that $K\backslash C$ is a repeller and that the evolutionary system $\mathcal{S}$ is both upper semicompact and lower semicontinuous. Then the invariance kernel $\mathrm{Inv}_{\mathcal{S}}(K,C)$ is a closed subset D between C and K satisfying*

$$\begin{cases} i) & D = \mathrm{Inv}_{\mathcal{S}}(D,C) \\ ii) & \overline{\complement D} = \mathrm{Capt}_{\mathcal{S}}(\overline{\complement D}, \overline{\complement K}) \end{cases}$$

Furthermore,

$$\mathrm{Int}(D) = \mathrm{Inv}_{\mathcal{S}}(\mathrm{Int}(K), \mathrm{Int}(D))$$

is invariant in $\mathrm{Int}(K)$ *outside* $\mathrm{Int}(D)$.

Proof — Let us consider the invariance kernel $D := \mathrm{Inv}_{\mathcal{S}}(K,C)$. It is the unique subset between C and K such that $D = \mathrm{Inv}_{\mathcal{S}}(D,C)$ and $D = \mathrm{Inv}_{\mathcal{S}}(K,D)$. Thanks to Lemma 3.3.2, the latter condition is equivalent to

$$\complement \mathrm{Inv}_{\mathcal{S}}(K,D) = \mathrm{Capt}_{\mathcal{S}}(\complement D, \complement K)$$

Since $\mathcal{S}$ is upper semicompact and since $\complement C \backslash \complement K = K\backslash C$ is a repeller, we deduce from Theorem 5.3.1 that

$$\overline{\complement D} \;=\; \overline{\mathrm{Capt}_{\mathcal{S}}(\complement D, \complement K)} \;\subset\; \mathrm{Capt}_{\mathcal{S}}(\overline{\complement D}, \overline{\complement K}) \;\subset\; \overline{\complement D}$$

and thus, that $\complement \overset{\circ}{D} = \mathrm{Capt}_{\mathcal{S}}(\complement \overset{\circ}{D}, \complement \overset{\circ}{K})$. By Lemma 3.3.2, we infer that $\mathrm{Int}(D) = \mathrm{Inv}_{\mathcal{S}}(\mathrm{Int}(K), \mathrm{Int}(D))$. ■

5.4 Characterization of Viability Kernels and Capture Basins

5.4.1 Subsets Viable outside a Target

We now provide a characterization of a subset D viable outside a target C in terms of local viability of $D\backslash C$:

Proposition 5.4.1. [***Characterization of Viable Subsets Outside a Target***] *Assume that $\mathcal{S}$ is upper semicompact. Let $C \subset D$ and D be closed subsets. The following* *conditions are equivalent:*

1. *D is viable outside C under $\mathcal{S}$ ($\mathrm{Viab}_{\mathcal{S}}(D, C) = D$ by Definition 1.1.3, p.16)*

2. *$D \setminus C$ is locally viable under $\mathcal{S}$ ($\mathrm{Exit}_{\mathcal{S}}(D) \subset C$ by Proposition 1.13.9, p.59)*

3. *The exit set of D is contained in the exit set of C ($\mathrm{Exit}_{\mathcal{S}}(D) \subset \mathrm{Exit}_{\mathcal{S}}(C)$)*

In particular, a closed subset D is viable under $\mathcal{S}$ if and only if its exit set is empty:

$$\mathrm{Viab}_{\mathcal{S}}(D) = D \text{ if and only if } \mathrm{Exit}_{\mathcal{S}}(D) = \emptyset$$

Proof

1. First, assume that $\mathrm{Viab}_{\mathcal{S}}(D, C) = D$ and derive that $D \setminus C$ is locally viable. Take $x_0 \in D \backslash C$ and prove that there exists an evolution $x(\cdot) \in \mathcal{S}(x_0)$ starting at x_0 viable in $D \backslash C$ on a nonempty interval. Indeed, since C is closed, there exists $\eta > 0$ such that $B(x_0, \eta) \cap C = \emptyset$, so that $x(t) \in B(x_0, \eta) \cap D \subset D \backslash C$ on some nonempty interval. This means that $\mathrm{Viab}_{\mathcal{S}}(D, C) \backslash C$ is locally viable.

2. Assume that $D \setminus C$ is locally viable and derive that $\mathrm{Exit}_{\mathcal{S}}(D) \subset \mathrm{Exit}_{\mathcal{S}}(C)$. Take $x^\sharp \in \mathrm{Exit}_{\mathcal{S}}(D)$, so that all evolutions starting from $x^\sharp$ leave D immediately. Such an element $x^\sharp$ belongs to C because, otherwise, since $D \backslash C$ is locally viable and C is closed, one could associate with $x^\sharp \in D \backslash C$ a persistent evolution $y^\sharp(\cdot) \in \mathcal{S}(x^\sharp)$ and $T > 0$ such that $y^\sharp(\tau) \in D \backslash C$ for all $\tau \in [0, T]$, so that $\tau_D^\sharp(x^\sharp) = T > 0$, contradicting the fact that $x^\sharp \in \mathrm{Exit}_{\mathcal{S}}(D)$. Hence, $x^\sharp \in \mathrm{Exit}_{\mathcal{S}}(D) \cap C$. Since evolutions starting from $x^\sharp \in C$ and leaving D immediately, leave also $C \subset D$ immediately, we infer that $x^\sharp$ belongs to $\mathrm{Exit}_{\mathcal{S}}(C)$.

3. Assume that $\mathrm{Exit}_{\mathcal{S}}(D) \subset \mathrm{Exit}_{\mathcal{S}}(C)$ and deduce that D is viable outside C under evolutions persistent in D. Let us take x in $D \backslash C$. Either $x \in \mathrm{Viab}_{\mathcal{S}}(D)$ and there exists an evolution $x(\cdot) \in \mathcal{S}(x)$ viable in D forever (and thus, persistent in K), or $x \in D \setminus \mathrm{Viab}_{\mathcal{S}}(D)$, and there exists a persistent evolution $x^\sharp(\cdot) \in \mathcal{S}^{D^\sharp}(x)$ which leaves D in finite time $\tau_D^\sharp(x)$ at $x^\sharp := x^\sharp(\tau_D^\sharp(x)) \in \mathrm{Exit}_{\mathcal{S}}(D)$ thanks to Proposition 1.13.13. Since $\mathrm{Exit}_{\mathcal{S}}(D) \subset \mathrm{Exit}_{\mathcal{S}}(C)$, $x^\sharp(\tau_D^\sharp(x)) \in C$ and $x^\sharp(\cdot)$ reaches C in finite time. Hence D is viable outside C by at least one evolution (the persistent one).

Taking $C = \emptyset$, we deduce the second statement of Proposition 5.4.2. ■

As a consequence, Proposition 5.4.2, Theorem 5.3.1 (guaranteeing that the viability kernels $\text{Viab}_{\mathcal{S}}(D,C)$ are closed) and Theorem 5.1.1 imply the following:

Proposition 5.4.2. [***Characterization of Viable Subsets Outside a Target***] *Assume that $\mathcal{S}$ is upper semicompact. Let $C \subset K$ and K be closed subsets.*

Then the viability kernel $\text{Viab}_{\mathcal{S}}(K,C)$ *of K with target C under $\mathcal{S}$ is the* largest closed subset $D \subset K$ *containing C such that $D \backslash C$ is locally viable.*

In particular, the viability kernel $\text{Viab}_{\mathcal{S}}(K)$ *of K is the largest closed viable subset contained in K.*

5.4.2 Relative Invariance

We characterize further isolated subsets in terms of backward invariance properties — discovered by Hélène Frankowska in her investigations of Hamilton-Jacobi equations associated with value functions of optimal control problems under state constraints. They play a crucial role for enriching the Characterization Theorem 5.1.1 stating that the viability kernel of an environment with a target is the smallest subset containing the target and isolated in this environment.

Definition 5.4.3. [***Relative Local Invariance***] *We shall say that a subset $C \subset K$ is* locally (backward) invariant relatively to K *under $\mathcal{S}$ if for every $x \in C$, there exist $T > 0$ such that all (backward) evolutions starting from x and viable in K on an interval $[0,T[$ are viable in C on the same interval $[0,T[$.*

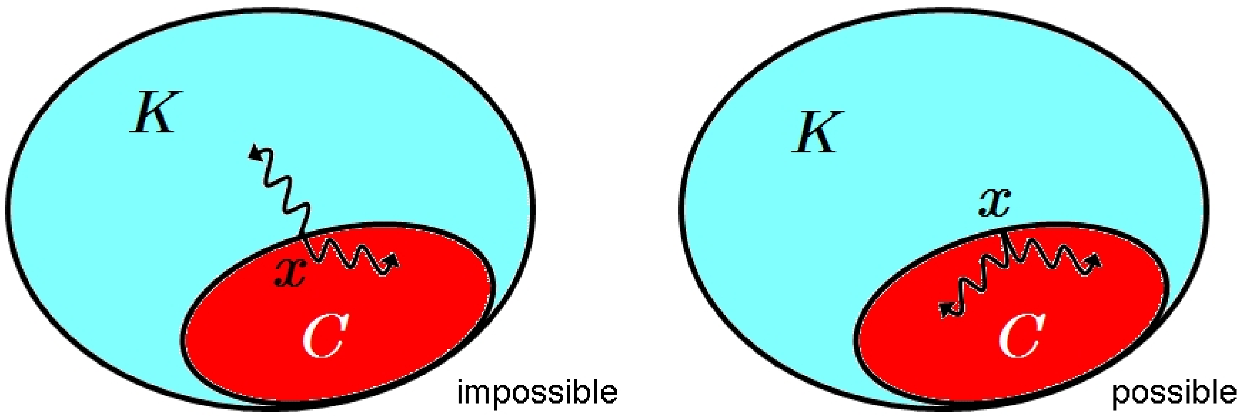


Figure 5.4: [**Local Backward Invariant Set**]

A subset $C \subset K$ is *locally backward invariant relatively to* K under $\mathcal{S}$ if for every $x \in C$, for every $t_0 \in]0, +\infty[$, for all evolutions $x(\cdot)$ arriving at x at time t_0 such that there exists $s \in [0, t_0[$ such that $x(\cdot)$ is viable in K on the interval $[s, t_0]$, then $x(\cdot)$ is viable in C on the same interval.

If K is itself locally (backward) invariant, any subset locally (backward) invariant relatively to K is locally (backward) invariant.

If $C \subset K$ is locally (backward) invariant relatively to K, then $C \cap \mathrm{Int}(K)$ is locally (backward) invariant.

From any $x \in C \cap \partial K$, all backward evolutions $x(\cdot) \in \mathcal{S}(x)$ (resp. $x(\cdot) \in \overleftarrow{\mathcal{S}}(x)$) satisfy

$$\begin{cases} \text{either } \exists\, T > 0 \text{ such that } \forall t \in [0, T],\ x(t) \in C \\ \text{or } \exists\, t_n \to 0+ \text{ such that } x(t_n) \in \complement K \end{cases}$$

Capture basins of targets viable in environments are backward invariants relatively to this environment:

Proposition 5.4.4. [***Relative Backward Invariance of Capture Basins***] *The capture basin* $\mathrm{Capt}_{\mathcal{S}}(K, C)$ *of a target* C *viable in the environment* K *is backward invariant relatively to* K*.*

Proof — We have to prove that for every $x \in \mathrm{Capt}_{\mathcal{S}}(K, C)$, every backward evolution viable in K on some time interval $[0, T]$ is actually viable in $\mathrm{Capt}_{\mathcal{S}}(K, C)$ on the same interval.

Since x belongs to $\mathrm{Capt}_{\mathcal{S}}(K,C)\backslash C$, there exists an evolution $z(\cdot) \in \mathcal{S}(x)$ and $S \geq 0$ such that $z(S) \in C$ and, for all $t \in [0,S]$, $x(t) \in K$. Let us consider any backward evolution $\overleftarrow{y}(\cdot) \in \overleftarrow{\mathcal{S}}(x)$ viable in K on some interval$[0,\overline{T}[$. We associate with any $T \in [0,\overline{T}[$ the evolution $\overrightarrow{x}_T(\cdot) \in \mathcal{S}(y(T))$ defined by

$$\overrightarrow{x}_T(t) \ := \begin{cases} \overleftarrow{y}(T-t) & \text{if} \quad t \in [0,T] \\ \overrightarrow{z}(t-T) & \text{if} \quad t \in [T, T+S] \end{cases}$$

starting at $y(T) \in K$. It is viable in K until it reaches C at time $T+S$. This means that $y(T)$ belongs to $\mathrm{Capt}_{\mathcal{S}}(K,C)$ for every $T \in [0,\overline{T}[$, i.e., that the backward evolution $\overleftarrow{y}(\cdot) \in \overleftarrow{\mathcal{S}}(x)$ is viable in $\mathrm{Capt}_{\mathcal{S}}(K,C)$ on the interval $[0,\overline{T}]$. ■

We deduce that a subset $C \subset K$ is locally backward invariant relatively to K if and only if K is the capture basin of C:

Theorem 5.4.5. [***Characterization of Relative Local Invariance***] *A subset $C \subset K$ is locally backward invariant relatively to K if and only if* $\mathrm{Capt}_{\mathcal{S}}(K,C) = C$.

Proof — First, Proposition 5.4.4, p.157 implies that whenever $\mathrm{Capt}_{\mathcal{S}}(K,C) = C$, C is backward invariant relatively to K. Conversely, assume that C is locally backward invariant relatively to K and consider $x \in \mathrm{Capt}_{\mathcal{S}}(K,C)\backslash C$: there exists a forward evolution denoted $\overrightarrow{x}(\cdot) \in \mathcal{S}(x)$ starting at x and viable in K until it reaches C at time $T > 0$ at $c = x(T)$. Let $\overleftarrow{z}(\cdot) \in \overleftarrow{\mathcal{S}}(x)$ be any backward evolution starting at x and viable in K on some interval $[0,T]$. We associate with it the function $\overleftarrow{y}(\cdot)$ defined by

$$\overleftarrow{y}(t) \ := \begin{cases} \overrightarrow{x}(T-t) & \text{if} \quad t \in [0,T] \\ \overleftarrow{z}(t-T) & \text{if} \quad t \geq T \end{cases}$$

Then $\overleftarrow{y}(\cdot) \in \overleftarrow{\mathcal{S}}(c)$ and is viable in K on the interval $[0,T]$. Since C is assumed to be locally backward invariant relatively to K, then $\overleftarrow{y}(t) \in C$ for all $t \in [0,T]$, and in particular $\overleftarrow{y}(T) = x$ belongs to C. We have obtained a contradiction since we assumed that $x \notin K$. Therefore $\mathrm{Capt}_{\mathcal{S}}(K,C)\backslash C = \emptyset$, i.e., $\mathrm{Capt}_{\mathcal{S}}(K,C) = C$. ■

As a consequence of Proposition 5.4.5, we obtain:

Proposition 5.4.6. [***Backward Invariance of the Complement of an Invariant***

Set] *A subset C is backward invariant under an evolutionary system $\mathcal{S}$ if and only if its complement $\complement C$ is invariant under $\mathcal{S}$.*

Proof — Applying Proposition 5.4.5 with $K := X$, we infer that C is backward invariant if and only if $C = \text{Capt}_{\mathcal{S}}(X, C)$, which is equivalent, by Lemma 3.3.2, to the statement that $\complement C = \text{Inv}_{\mathcal{S}}(\complement C, \emptyset) =: \text{Inv}_{\mathcal{S}}(\complement C)$ is invariant. ■

5.4.3 Isolated Subsets

The following Lemma is useful:

Lemma 5.4.7. [***Isolated Subsets***] *Let D and K be two subsets such that $D \subset K$. Then the following properties are equivalent:*

1. *D is isolated in K under $\mathcal{S}$:* $\text{Viab}_{\mathcal{S}}(K, D) = D$,
2. $\text{Viab}_{\mathcal{S}}(K) = \text{Viab}_{\mathcal{S}}(D)$ *and* $\text{Capt}_{\mathcal{S}}(K, D) = D$,
3. *$K \backslash D$ is a repeller and* $\text{Capt}_{\mathcal{S}}(K, D) = D$.

Proof — Assume that D is isolated in K. This amounts to writing that,

1. by definition,
$$D = \text{Viab}_{\mathcal{S}}(K, D) = \text{Viab}_{\mathcal{S}}(K) \cup \text{Capt}_{\mathcal{S}}(K, D)$$
and thus, equivalently, that $\text{Capt}_{\mathcal{S}}(K, D) = D$ and $\text{Viab}_{\mathcal{S}}(K) \subset D$. Since $D \subset K$, inclusion $\text{Viab}_{\mathcal{S}}(K) \subset D$ is equivalent to $\text{Viab}_{\mathcal{S}}(K) = \text{Viab}_{\mathcal{S}}(D)$.
2. by formula (1.23),
$$D = \text{Viab}_{\mathcal{S}}(K, D) = \text{Viab}_{\mathcal{S}}(K \backslash D) \cup \text{Capt}_{\mathcal{S}}(K, D)$$
and thus, equivalently, that $\text{Viab}_{\mathcal{S}}(K \backslash D) = \emptyset$ and $\text{Capt}_{\mathcal{S}}(K, D) = D$, because $D \cap \text{Viab}_{\mathcal{S}}(K \backslash D) = \emptyset$. ■

We derive the following characterization:

Theorem 5.4.8. [***Characterization of Isolated Subsets***] *Let us consider a closed subset $D \subset K$. Then D is isolated in K by $\mathcal{S}$ if and only if*

1. *D is locally backward invariant relatively to K,*
2. *either $K \backslash D$ is a repeller or $\mathrm{Viab}_{\mathcal{S}}(K) = \mathrm{Viab}_{\mathcal{S}}(D)$.*

We provide now another characterization of isolated subsets involving complements:

Lemma 5.4.9. [***Complement of an Isolated Subset***] *Let us assume that the evolutionary system $\mathcal{S}$ is lower semicontinuous and that K and $D \subset K$ are closed.*

1. *If $D = \mathrm{Capt}_{\mathcal{S}}(K, D)$, then $\mathrm{Int}(K) \backslash \mathrm{Int}(D)$ is locally invariant relatively to the interior $\mathrm{Int}(K)$ of K.*
2. *Conversely, if $\mathrm{Int}(K)$ is backward invariant and if $\mathrm{Int}(K) \backslash \mathrm{Int}(D)$ is locally invariant relatively to the interior $\mathrm{Int}(K)$ of K, then*

$$\mathrm{Capt}_{\mathcal{S}}(K, \overline{\mathrm{Int}(D)}) = \overline{\mathrm{Int}(D)}$$

Proof — Lemma 3.3.2 implies that $\mathrm{Capt}_{\mathcal{S}}(K, D) = D$ if and only if

$$\complement D = \mathrm{Inv}_{\mathcal{S}}(\complement D, \complement K)$$

Since $\mathcal{S}$ is assumed to be lower semicontinuous, we deduce from Theorem 5.2.8 that

$$\complement(\mathrm{Int}(D)) = \overline{\complement D} = \overline{\mathrm{Inv}_{\mathcal{S}}(\complement D, \complement K)} \subset \mathrm{Inv}_{\mathcal{S}}(\overline{\complement D}, \overline{\complement K}) = \mathrm{Inv}_{\mathcal{S}}(\complement(\mathrm{Int}(D)), \complement(\mathrm{Int}(K))) \subset \complement(\mathrm{Int}(D))$$

so that the closure of the complement of D is invariant outside the closure of the complement of K. Observe that, taking the complements, Lemma 3.3.2 implies that $\mathrm{Int}(D) = \mathrm{Capt}_{\mathcal{S}}(\mathrm{Int}(K), \mathrm{Int}(D))$.

Therefore, for any $x \in \mathrm{Int}(K) \backslash \mathrm{Int}(D)$, for any $x(\cdot) \in \mathcal{S}(x)$ such that $x(s) \in \mathrm{Int}(K)$ for every $s \in [0, t]$, then $x(s) \in \mathrm{Int}(K) \backslash \mathrm{Int}(D)$ for every $s \in [0, t]$: This means that $\mathrm{Int}(K) \backslash \mathrm{Int}(D)$ is locally invariant relatively to K.

Conversely, assume that $\mathrm{Int}(K) \backslash \mathrm{Int}(D)$ is locally invariant relatively to $\mathrm{Int}(K)$. If there would exist $x \in \mathrm{Capt}_{\mathcal{S}}(\mathrm{Int}(K), \mathrm{Int}(D)) \backslash \mathrm{Int}(D)$, then there exist an evolution $x(\cdot) \in \mathcal{S}(x)$ and a finite time t such that $x(t) \in \mathrm{Int}(D)$ and $x(s) \in \mathrm{Int}(K)$ for every $s \in [0, t]$, a contradiction.

Therefore

$$\text{Capt}_{\mathcal{S}}(\text{Int}(K), \text{Int}(D)) \subset \text{Int}(D) \tag{5.9}$$

Assume furthermore that the evolutionary system $\mathcal{S}$ is lower semi-continuous, that K is closed and that $\text{Int}(K)$ is backward invariant. Theorem 5.3.6 implies that

$$\text{Capt}_{\mathcal{S}}(K, \overline{\text{Int}(D)}) \subset \overline{\text{Capt}_{\mathcal{S}}(\text{Int}(K), \text{Int}(D))} \subset \overline{\text{Int}(D)} \;\; \blacksquare$$

5.4.4 The Second Fundamental Characterization Theorem

Putting together the characterizations of viable subsets and isolated subsets, we reformulate Theorem 5.1.1 characterizing viability kernels with targets in the following way:

Theorem 5.4.10. [***Characterization of Viability Kernels with Targets***] *Let us assume that $\mathcal{S}$ is upper semicompact and that the subsets $C \subset K$ and K are closed. The viability kernel* $\text{Viab}_{\mathcal{S}}(K, C)$ *of a subset K with target C under $\mathcal{S}$ is the* **unique** *closed subset satisfying $C \subset D \subset K$ and*

$$\begin{cases} (i) & D \backslash C \text{ is } \textit{locally viable} \text{ under } \mathcal{S} \\ (ii) & D \textit{ is locally backward invariant relatively to } K \text{ under } \mathcal{S} \\ (iii) & K \backslash D \text{ is a } \textit{repeller} \text{ under } \mathcal{S} \text{ or } \text{Viab}_{\mathcal{S}}(K) = \text{Viab}_{\mathcal{S}}(D). \end{cases} \tag{5.10}$$

Theorem 5.4.10 implies that when the target C is empty, the above theorem implies a characterization of viability kernels:

Theorem 5.4.11. [***Characterization of Viability Kernels***] *Let us assume that $\mathcal{S}$ is upper semicompact and that the subset K is closed. The viability kernel* $\text{Viab}_{\mathcal{S}}(K)$ *of a subset K under $\mathcal{S}$ is the* **unique** *closed subset satisfying $C \subset D \subset K$ and*

$$\begin{cases} (i) & D \text{ is } \textit{viable} \text{ under } \mathcal{S} \\ (ii) & D \textit{ is locally backward invariant relatively to } K \text{ under } \mathcal{S} \\ (iii) & K \backslash D \text{ is a } \textit{repeller} \text{ under } \mathcal{S} \text{ or } \text{Viab}_{\mathcal{S}}(K) = \text{Viab}_{\mathcal{S}}(D). \end{cases} \tag{5.11}$$

Theorem 5.4.10 implies that when $K\backslash C$ is a repeller, the above theorem implies a characterization of the viable-capture basins:

Theorem 5.4.12. [***Characterization of Capture Basins***] *Let us assume that $\mathcal{S}$ is upper semicompact and that a closed subset $C \subset K$ satisfies property*

$$\mathrm{Viab}_{\mathcal{S}}(K\backslash C) = \emptyset \tag{5.12}$$

Then the viable-capture basin $\mathrm{Capt}_{\mathcal{S}}(K, C)$ is the **unique** *closed subset D satisfying $C \subset D \subset K$ and*

$$\left\{ \begin{array}{ll} (i) & D\backslash C \text{ is } \textit{locally viable} \text{ under } \mathcal{S} \\ (ii) & D \textit{ is locally backward invariant relatively to } K \text{ under } \mathcal{S} \end{array} \right. \tag{5.13}$$

It is also convenient to reformulate Theorem 5.4.12 characterizing backward capture basins of targets as subsets D crossed by evolutions entering it through $\mathrm{Exit}_{\overleftarrow{\mathcal{S}}}(C)$ and leaving it through $\mathrm{Exit}_{\mathcal{S}}(K)$:

Proposition 5.4.13. [***Characterization of Backward Capture Basins***] *Let us assume that $\mathcal{S}$ is upper semicompact and that the subsets $C \subset K$ and K are closed. If K is a backward repeller, the backward capture basin $\mathrm{Capt}_{\overleftarrow{\mathcal{S}}}(K, C)$ of a subset K with target C is the* **unique** *closed subset D between C and K of elements x through which*

1. *passes at least one evolution starting from $\mathrm{Exit}_{\overleftarrow{\mathcal{S}}}(D) \subset \mathrm{Exit}_{\overleftarrow{\mathcal{S}}}(C)$ and viable in D until it reaches x,*
2. *all evolutions starting from x are viable in D until they leave K through $\mathrm{Exit}_{\mathcal{S}}(K)$.*

Proof — More precisely, we shall prove that the capture basin is the unique subset D of elements $x \in K$ satisfying

$$\left\{ \begin{array}{ll} (i) & \textbf{there exist } x^\star \in \mathrm{Exit}_{\overleftarrow{\mathcal{S}}}(D) \subset \mathrm{Exit}_{\overleftarrow{\mathcal{S}}}(C),\ t^\star \geq 0 \text{ and an evolution} \\ & x(\cdot) \in \mathcal{S}(x^\star) \textbf{ viable in } D \textbf{ on } [0, t^\star] \text{ such that } x(t^\star) = x \\ (ii) & \textbf{all evolutions} \text{ starting from } x \text{ at time } t^\star \text{ are viable in } D \text{ on } [t^\star, \tau_K(x(\cdot))] \end{array} \right. \tag{5.14}$$

Indeed, by definition, $\text{Capt}_{\overleftarrow{\mathcal{S}}}(D,C) = D$ if and only if from any $x \in D$ starts at least one backward evolution $\overleftarrow{x}(\cdot) \in \overleftarrow{\mathcal{S}}(x)$ persistent in D and $t^\star := \overleftarrow{\tau}^\sharp_D(x) < +\infty$ such that

$$x^\star := \overleftarrow{x}(t^\star) \in \text{Exit}_{\overleftarrow{\mathcal{S}}}(D) \subset \text{Exit}_{\overleftarrow{\mathcal{S}}}(C)$$

and for all $t \in [0,t^\star]$, $\overleftarrow{x}(t) \in D$. Setting $x(t) := \overleftarrow{x}(t^\star - t)$ when $t \in [0,t^\star]$, this is equivalent to (5.14)(i).

Theorem 5.4.5 states that $\text{Capt}_{\overleftarrow{\mathcal{S}}}(K,D) = D$ if and only if from any $x \in D$, all evolutions $\overrightarrow{x}(\cdot) \in \mathcal{S}(x)$ starting from x viable in K are viable in D. This amounts to saying that $\overrightarrow{x}(\cdot)$ is viable in D on $[0, \tau_K(\overrightarrow{x}(\cdot))]$.

Setting $x(t) := \overrightarrow{x}(t - t^\star)$ whenever $t \in [t^\star, \tau_K(x(\cdot))]$, this is equivalent to (5.14)(ii). ■

We deduce from Lemma 5.4.9 another characterization of capture basins that provide existence and uniqueness of viscosity solutions to some Hamilton-Jacobi-Bellman equations:

Theorem 5.4.14. [***"Viscosity" Characterization of Capture Basins***] *Assume that the evolutionary system $\mathcal{S}$ is both upper semicompact and lower semicontinuous, that K is closed and that $\text{Int}(K)$ is backward invariant, that $\text{Viab}_{\mathcal{S}}(K\backslash C) = \emptyset$ and that $\overline{\text{Int}(C)} = C$.*

*Then the capture basin $\text{Capt}_{\mathcal{S}}(K,C)$ is the **unique** subset D between C and K satisfying*

$$\begin{cases} (i) & D\backslash C \text{ is } \textit{locally viable} \text{ under } \mathcal{S}, \\ (ii) & \text{Int}(K)\backslash\text{Int}(D) \text{ is locally invariant under } \mathcal{S}. \end{cases} \tag{5.15}$$

Furthermore, the capture basin of C viable in K is equal to the closure of its interior:

$$\text{Capt}_{\mathcal{S}}(K,C) = \overline{\text{Int}(\text{Capt}_{\mathcal{S}}(K,C))}$$

Proof — Since $\text{Viab}_{\mathcal{S}}(K\backslash C) = \emptyset$, the viability kernel and the capture basin are equal. By Theorem 5.1.1,the capture basin is the unique subset D between C and K such that

1. the largest subset $D \subset K$ viable outside C,

2. the smallest subset $D \supset C$ such that $\text{Capt}_{\mathcal{S}}(K,D) = D$.

The evolutionary system being upper semicompact, the first condition amounts to saying that $D\backslash C$ is locally viable. Since the evolutionary system is also lower semicontinuous, we deduce from the first statement of Lemma 5.4.9 that the second property implies that $\text{Int}(K)\backslash\text{Int}(D)$ is locally invariant relatively to the interior $\text{Int}(K)$. Hence the capture basin satisfies properties (5.15).

Conversely, let D satisfy those properties (5.15), we infer from Proposition 5.4.2 and the second statement of Lemma 5.4.9 that

$$D \subset \mathrm{Capt}_{\mathcal{S}}(K, C) = \mathrm{Capt}_{\mathcal{S}}(K, \overline{\mathrm{Int}(C)}) \subset \mathrm{Capt}_{\mathcal{S}}(K, \overline{\mathrm{Int}(D)}) \subset \overline{\mathrm{Int}(D)} \subset D \;\blacksquare$$

Remark: We shall see that whenever the environment $K := \mathcal{E}p(\mathbf{b})$ and the target $C := \mathcal{E}p(c)$ are epigraphs of functions $\mathbf{b} \leq \mathbf{c}$, the capture basin under adequate dynamical system is itself the epigraph of a function $\mathbf{v}$. Theorem 5.4.14, p.163 implies that $\mathbf{v}$ is a viscosity solution to an Hamilton-Jacobi-Bellman equation. $\blacksquare$

5.5 Characterization of Invariance Kernels

Proposition 5.5.1. [***Characterization of Invariant Subsets Outside a Target***] *Assume that $\mathcal{S}$ is upper lower semicontinuous. Let $C \subset K$ be closed subsets.*

Then the invariance kernel $\mathrm{Inv}_{\mathcal{S}}(K, C)$ of K with target C under $\mathcal{S}$ is the largest closed subset *$D \subset K$ containing C such that $D \backslash C$ is locally invariant.*

In particular, K is invariant outside C if and only if $K \backslash C$ is locally invariant.

Proof — First, we have to check that if $D \supset C$ is invariant outside C, then $D \backslash C$ is locally invariant: Take $x_0 \in D \backslash C$ and prove that all evolutions $x(\cdot) \in \mathcal{S}$ starting at x_0 are viable in $D \backslash C$ on a nonempty interval. Indeed, since C is closed, there exists $\eta > 0$ such that $B(x_0, \eta) \cap C = \emptyset$, so that $x(t) \in B(x_0, \eta) \cap D \subset D \backslash C$ on some nonempty interval.

In particular, $\mathrm{Inv}_{\mathcal{S}}(K, C) \backslash C$ is locally invariant and the invariance kernel $\mathrm{Inv}_{\mathcal{S}}(K, C)$ of K with target C under $\mathcal{S}$ is closed by Theorem 5.3.3.

Let us prove now that any subset D between C and K such that $D \backslash C$ is locally invariant is contained in the invariance kernel $\mathrm{Inv}_{\mathcal{S}}(K, C)$ of K with target C under $\mathcal{S}$.

Since $C \subset \mathrm{Inv}_{\mathcal{S}}(K, C)$, let us pick any x in $D \backslash C$ and show that it belongs to $\mathrm{Inv}_{\mathcal{S}}(K, C)$. Let us take any evolution $x(\cdot) \in \mathcal{S}(x)$. Either it is viable in D forever or, if not, leaves D in finite time $\tau_D(x(\cdot))$ at $\overline{x} := x(\tau_D(x(\cdot)))$: there exists a sequence $t_n \geq \tau_D(x(\cdot))$ converging to $\tau_D(x(\cdot))$ such that $x(t_n) \notin D$. Actually, this element $\overline{x}$ belongs to C. Otherwise, since $D \backslash C$ is locally invariant, this evolution remains in D in some nonempty interval $[\tau_D(x(\cdot))), T]$, a contradiction. $\blacksquare$

5.6 The Barrier Property

Roughly speaking, a subset enjoys the barrier property if all locally viable evolutions starting from its boundary are viable on its boundary, so that no evolution can enter the interior of set.

For that purpose, we need to define the concept of boundary:

Definition 5.6.1. [***Boundaries***] *Let $C \subset K \subset X$ be two subsets of X. The subsets*

$$\partial_K C := \overline{C} \cap \overline{K \backslash C} \ \& \ \overset{\circ}{\partial}_K C := C \cap \overline{K \backslash C}$$

are called respectively the boundary and the pre-boundary of the subset C relatively to K. When $K := X$, we set

$$\partial C := \overline{C} \cap \overline{\complement C} \ \& \ \overset{\circ}{\partial} C := C \cap \overline{\complement C}$$

In other words, the interior of D and its pre-boundary form a partition of $D = \mathrm{Int}(D) \cup \overset{\circ}{\partial} D$. Pre-boundaries are useful because of the following property:

Lemma 5.6.2. [***Pre-boundary of an intersection with an open set***] *Let $\Omega \subset X$ be an open subset and $D \subset X$ be a subset. Then*

$$\overset{\circ}{\partial} (\Omega \cap D) = \Omega \cap \overset{\circ}{\partial} D$$

In particular, if $C \subset D$ is closed, then $\mathrm{Int}(D) \setminus C = \mathrm{Int}(D \setminus C)$ *and* $\overset{\circ}{\partial} (D \setminus C) = \overset{\circ}{\partial} (D) \setminus C$.

Proof — Indeed, $D = \mathrm{Int}(D) \cup \overset{\circ}{\partial} D$ being a partition of D, we infer that $D \cap \Omega = \mathrm{Int}(D \cap \Omega) \cup \overset{\circ}{\partial} D \cap \Omega$ being still a partition. By definition, $D \cap \Omega = \mathrm{Int}(D \cap \Omega) \cup \overset{\circ}{\partial} (D \cap \Omega)$ is another partition of $D \cap \Omega$. Since Ω is open, $\mathrm{Int}(D \cap \Omega) = \mathrm{Int}(D) \cap \mathrm{Int}(\Omega) = mbox\mathrm{Int}(D) \cap \Omega$, so that $\overset{\circ}{\partial} (D \cap \Omega) = \Omega \cap \overset{\circ}{\partial} D$. ■

Definition 5.6.3. [***Barriers and Semi-Permeability***] *Let $D \subset X$ be a subset and $\mathcal{S}$ be an evolutionary system. We shall say that D enjoys the local barrier property if its pre-boundary $\overset{\circ}{\partial} D$ is locally invariant with respect to D itself : Starting from any $x \in \overset{\circ}{\partial} D$,* *all evolutions viable in D on some time interval $[0,T[$ are viable in $\overset{\circ}{\partial} D$ on $[0,T[$. This is a* semi-permeability property *of D, since no evolutions can enter the interior of D from the boundary (whereas evolutions may leave D).*

This is very important in terms of interpretation. Viability of a subset D having the barrier property is indeed a very fragile property, which cannot be reestablished from the outside: In other words, starting from the pre-boundary, *love it or leave it ...*

We deduce from Theorem 5.4.5, p.158 that a subset D enjoys the barrier property if and only if its interior is backward invariant:

Proposition 5.6.4. [***Backward Invariance of the interior and Barrier Property***] *A subset D enjoys the barrier property if and only if its interior* $\mathrm{Int}(D)$ *is backward invariant.*

Proof — Theorem 5.4.5, p.158 states that the pre-boundary $\overset{\circ}{\partial} D \subset D$ is locally invariant relatively to D if and only if $\mathrm{Capt}_{\overleftarrow{\mathcal{S}}}(D, \overset{\circ}{\partial} D) = \overset{\circ}{\partial} D$. Therefore, from every $x \in \mathrm{Int}(D) = D \setminus \overset{\circ}{\partial} D = D \setminus \mathrm{Capt}_{\overleftarrow{\mathcal{S}}}(D, \overset{\circ}{\partial} D)$, all backward evolutions are viable in $\mathrm{Int}(D) = D \setminus \overset{\circ}{\partial} D$ as long as they are viable in D. Such evolutions remain always in $\mathrm{Int}(D)$ because they can never reach $x(t) \in \overset{\circ}{\partial} D$ at some finite time t. ■

Theorem 5.6.5. [***The Barrier Property of Viability Kernels***] *Assume that K and $C \subset K$ are closed and that the evolutionary system $\mathcal{S}$ is lower semicontinuous. Then the intersection* $\mathrm{Viab}_{\mathcal{S}}(K,C) \cap \mathrm{Int}(K \setminus C)$ *of the viability kernel of K with target $C \subset K$ under $\mathcal{S}$ with the intersection of the interior of $K \setminus C$ enjoys the* barrier property.

In particular, the intersection $\mathrm{Viab}_{\mathcal{S}}(K) \cap \mathrm{Int}(K)$ *enjoys the barrier property. If* $\mathrm{Viab}_{\mathcal{S}}(K,C) \subset \mathrm{Int}(K)$*, then* $\mathrm{Viab}_{\mathcal{S}}(K,C) \setminus C$ *enjoys the barrier property and, if* $\mathrm{Viab}_{\mathcal{S}}(K) \subset \mathrm{Int}(K)$*, then* $\mathrm{Viab}_{\mathcal{S}}(K)$ *enjoys the barrier property.*

Proof — Since C is closed, Lemma 5.6.2, p.165 states that the pre-boundary of the intersection $\mathrm{Viab}_{\mathcal{S}}(K,C) \cap \mathrm{Int}(K \setminus C)$ is equal to the intersection of $\mathrm{Int}(K \setminus C)$ with the pre-boundary $\overset{\circ}{\partial} (\mathrm{Viab}_{\mathcal{S}}(K,C))$ of the viability kernel.

Let x belong to $\mathrm{Int}(K \setminus C) \cap \overset{\circ}{\partial} (\mathrm{Viab}_{\mathcal{S}}(K,C))$ and $x(\cdot) \in \mathcal{S}(x)$ be a solution viable in K forever ($\varpi_C(x(\cdot)) = +\infty$) or until it reaches C at finite time $\varpi_C(x(\cdot)) < +\infty$. Since all evolutions starting at x, and thus, this evolution $x(\cdot) \in \mathcal{S}(x)$, are viable in $\overline{\complement \, \mathrm{Viab}_{\mathcal{S}}(K,C)}$ as long as $x(\cdot)$ are viable in K. Hence $x(\cdot)$ is viable in $\overset{\circ}{\partial} \, \mathrm{Viab}_{\mathcal{S}}(K,C)$ as long as $x(\cdot)$ is viable in $\mathrm{Int}(K \backslash C)$.

If $\mathcal{S}$ is upper semicompact, the viability kernel is closed, so that its pre-boundary coincides with its boundary. ■

Remark: Barrier Property The "barrier property" plays an important role in control theory and the theory of differential games. Marc Quincampoix made the link between this property and the boundary of the viability kernel: Every solution starting from the boundary of the viability kernel can either remain in the boundary or leave the viability kernel, or equivalently, no solution starting from outside the viability kernel can cross its boundary: such solutions can only remain on the boundary of the viability kernel, or leave it. ■

Theorem 5.6.6. [***Backward Invariance of the Interior of Viability Kernels***] *Assume that K and $C \subset K$ are closed and that the evolutionary system $\mathcal{S}$ is lower semicontinuous. Then the interior of* $\mathrm{Int}(\mathrm{Viab}_{\mathcal{S}}(K,C)) \setminus C)$ *of the viability kernel of K with target $C \subset K$ under $\mathcal{S}$ outside C is backward invariant.*

In particular, the interior $\mathrm{Int}(\mathrm{Viab}_{\mathcal{S}}(K))$ *is backward invariant.*

We investigate now the viability property of invariance kernels:

Lemma 5.6.7. [***Complement of a Separated Subset***] *Let us assume that the evolutionary system $\mathcal{S}$ is upper semicompact and that a closed subset $D \subset K$ is separated from K. Then* $\mathrm{Int}(K \setminus D) \backslash \mathrm{Int}(D)$ *is locally viable under $\mathcal{S}$. In particular, if $C \subset K$ is closed,* $\mathrm{Int}(K) \backslash \mathrm{Int}(\mathrm{Inv}_{\mathcal{S}}(K,C))$ *is locally viable.*

Proof — Let $x \in \mathrm{Int}(K) \backslash \mathrm{Int}(D)$ be given and $x_n \in \mathrm{Int}(K) \backslash D$ converge to x. Since $D = \mathrm{Inv}_{\mathcal{S}}(K,D)$ by assumption, for any n, there exists $x_n(\cdot) \in \mathcal{S}(x_n)$ such that

$$T_n := \varpi_{\partial K}(x_n(\cdot)) \ \leq \ \tau_K(x_n(\cdot)) \leq \varpi_D(x_n(\cdot))$$

because $x_n \in K\backslash D$ and $\varpi_{\partial K}(x_n(\cdot)) \leq \tau_K(x_n(\cdot)) < +\infty$. Therefore, for any $t < \varpi_{\partial K}(x_n(\cdot))$, $x_n(t) \in \mathrm{Int}(K) \setminus D$.

Since $\mathcal{S}$ is upper semicompact, a subsequence (again denoted by) $x_n(\cdot)$ converges to some $x(\cdot) \in \mathcal{S}(x)$. Since the functional $\varpi_{\partial K}$ is lower semicontinuous, we know that for any $t < \varpi_{\partial K}(x(\cdot))$, we have $t < \varpi_{\partial K}(x_n(\cdot))$ for n large enough. Consequently, $x_n(t) \in \complement D$, and, passing to the limit, we infer that for any $t < \varpi_{\partial K}(x(\cdot))$, $x(t) \in \overline{\complement D}$. This solution is thus locally viable in $\mathrm{Int}(K)\backslash\mathrm{Int}(D)$. ■

The boundary of the invariance kernel is locally viable:

Theorem 5.6.8. [***Local Viability of the Boundary of an Invariance Kernel***] *If $C \subset K$ and K are closed and if $\mathcal{S}$ is upper semicompact, then, for every $x \in \overset{\circ}{\partial} (\mathrm{Inv}_{\mathcal{S}}(K, C)) \cap \mathrm{Int}(K \setminus C)$, there exists at least one solution $x(\cdot) \in \mathcal{S}(x)$ locally viable in*

$$\overset{\circ}{\partial}\, \mathrm{Inv}_{\mathcal{S}}(K, C) \cap \mathrm{Int}(K\backslash C)$$

Proof — Let x belong to $\overset{\circ}{\partial}\, \mathrm{Inv}_{\mathcal{S}}(K, C) \cap \mathrm{Int}(K)$. Lemma 5.6.7, p.167 states there exists an evolution $x(\cdot)$ viable in $\mathrm{Int}(K) \cap \overline{\complement}\, \partial_K(\mathrm{Inv}_{\mathcal{S}}(K, C)$ since the invariance kernel is separated from K. Since x belongs to the invariance kernel, it is viable in $\mathrm{Inv}_{\mathcal{S}}(K, C)$ until it reaches the target C, and thus viable in $\overset{\circ}{\partial}\, \mathrm{Inv}_{\mathcal{S}}(K, C)$ as long as it is viable in the interior of $K \setminus C$. ■

The boundary of the viability kernel can be characterized as the viability kernel of the complement of a target:

Theorem 5.6.9. [***Characterization of the Boundary of a Viability Kernel***] *Let $C \subset K$ be a nonempty closed target contained in K. Then the two following conditions are equivalent:*

1. *the viability kernel of $K\backslash C$ is equal to the boundary of the viability kernel of K:*

$$\partial \mathrm{Viab}_{\mathcal{S}}(K) = \mathrm{Viab}_{\mathcal{S}}(K\backslash C)$$

2. *the boundary $\partial \mathrm{Viab}_{\mathcal{S}}(K)$ is viable under $\mathcal{S}$ and the interior $\mathrm{Int}(\mathrm{Viab}_{\mathcal{S}}(K))$ of the*

viability kernel of K under $\mathcal{S}$ absorbs $C \cap \mathrm{Int}(\mathrm{Viab}_{\mathcal{S}}(K))$:

$$\begin{cases} (i) & \partial \mathrm{Viab}_{\mathcal{S}}(K) = \mathrm{Viab}_{\mathcal{S}}(\partial \mathrm{Viab}_{\mathcal{S}}(K)) \\ (ii) & \mathrm{Int}(\mathrm{Viab}_{\mathcal{S}}(K)) = \mathrm{Abs}_{\mathcal{S}}\mathrm{Int}(\mathrm{Viab}_{\mathcal{S}}(K)), C \cap \mathrm{Int}(\mathrm{Viab}_{\mathcal{S}}(K))) \end{cases}$$

Proof — The fundamental Theorem 5.1.1 states that $\mathrm{Viab}_{\mathcal{S}}(K \setminus C)$ is the unique subset $D \subset K \setminus C$ such that $\mathrm{Viab}_{\mathcal{S}}(D) = D$ and $\mathrm{Viab}_{\mathcal{S}}(K \setminus C, D) = D$.

Therefore, to say that $D = \partial \mathrm{Viab}_{\mathcal{S}}(K)$ amounts to saying that $C \cap \partial K = \emptyset$, that $\partial \mathrm{Viab}_{\mathcal{S}}(K) = \mathrm{Viab}_{\mathcal{S}}(\partial \mathrm{Viab}_{\mathcal{S}}(K))$ and that

$$\mathrm{Viab}_{\mathcal{S}}(K \setminus C, \partial \mathrm{Viab}_{\mathcal{S}}(K)) = \partial \mathrm{Viab}_{\mathcal{S}}(K)$$

By taking complements, this equation can be written

$$\complement \partial \mathrm{Viab}_{\mathcal{S}}(K) = \mathrm{Abs}_{\mathcal{S}}(\complement \partial \mathrm{Viab}_{\mathcal{S}}(K), C \cup \complement K)$$

Consider now $x \in \complement \partial \mathrm{Viab}_{\mathcal{S}}(K)$. Then three possibilities can occur:

- $x \in \complement K$, which is contained in $\mathrm{Abs}_{\mathcal{S}}(\partial \mathrm{Viab}_{\mathcal{S}}(K), C \cup \complement K)$,
- $x \in K \setminus \mathrm{Viab}_{\mathcal{S}}(K)$, and in this case, all evolutions are viable in the complement of the viability kernel until reach the complement of K
- $x \in \mathrm{Int}(\mathrm{Viab}_{\mathcal{S}}(K))$. The above equation means that all evolutions $x(\cdot) \in \mathcal{S}(x)$ are viable in the complement of the boundary of the viability kernel until they reach either C or $\complement K$. Starting from the interior of $\mathrm{Viab}_{\mathcal{S}}(K)$, they cannot reach the boundary $\partial \mathrm{Int}(\mathrm{Viab}_{\mathcal{S}}(K))$ in finite time, because, being viable, this would imply that x would belong to $\partial \mathrm{Int}(\mathrm{Viab}_{\mathcal{S}}(K))$. Hence they are viable in the interior of the viability kernel of K until they reach C in finite time.

Knowing that the boundary of the viability kernel of K is viable, we have proved that equation

$$\mathrm{Viab}_{\mathcal{S}}(K \setminus C, \partial \mathrm{Viab}_{\mathcal{S}}(K)) = \partial \mathrm{Viab}_{\mathcal{S}}(K)$$

boils down to equation

$$\mathrm{Int}(\mathrm{Viab}_{\mathcal{S}}(K)) = \mathrm{Abs}_{\mathcal{S}}\mathrm{Int}(\mathrm{Viab}_{\mathcal{S}}(K)), C \cap \mathrm{Int}(\mathrm{Viab}_{\mathcal{S}}(K)))$$

This completes the proof. ■

5.7 Tangential Conditions

The important task is to characterize the subsets viable or invariant under a differential inclusion. To be of value, this task must be done without checking the existence of viable solutions for each initial state, but rather by checking whether some conditions relating the geometry of the environment K and the right hand side of the differential inclusion are satisfied. Allowing us to avoid solving the system, the verification that a constrained subset is viable is a much easier task.

An immediate intuitive idea jumps to the mind: at each point on the boundary of the viability set, where the viability of the system is at stake, there should exist a velocity which in some sense is *tangent* to the viability domain for allowing the solution to bounce back in the environment and remain inside it. This is, in essence, what the Viability Theorem states. But, first, the mathematical implementation of the concept of tangency inherited from Fermat must be made.

We cannot be content with viability sets that are smooth manifolds (such as spheres, which have no interior), because inequality constraints would thereby be ruled out (as for balls, that possess distinct boundary). Furthermore, we are no longer free of choosing environments and targets, some of them being provided as results of other problems (such as viability kernels and capture basins of other sets). So, we need to "implement" the concept of a direction v tangent to any subset K at $x \in K$, which should mean that starting from x in the direction v, "we do not go too far" from K.

To convert this intuition into a rigorous mathematical definition, we shall choose from among the many ways there are to translate what it means to be "not too far" the one independently suggested in the beginning of the 1930's by Georges Bouligand and Francesco Severi.

Definition 5.7.1. [**Contingent (Tangent) Cone to a Subset**] *Let $K \subset X$ be a subset and $x \in K$ an element of K. A direction v is* contingent *(or, more simply, "tangent") to K at $x \in K$ if it is a limit of a sequence of directions v_n such that $x + h_n v_n$ belongs to K for some sequence $h_n \to 0+$. The collection of such contingent directions constitutes a closed cone $T_K(x)$, called the* contingent cone *to K at x, or more simply,* tangent cone.

Except if K is a smooth manifold, the set of tangent vectors is no longer a vector-space, but this discomfort is compensated by advances in set-valued analysis providing a calculus of tangent cones allowing us to compute them.

Tangent cones allow us to "differentiate" viable evolutions: Indeed, we observe readily the following property.

Lemma 5.7.2. [**Derivatives of Viable Evolutions**] *Let $x(\cdot) : t \in \mathbb{R}_+ \mapsto x(t) \in X$ be a differentiable evolution viable in K on an open interval $\mathbb{I}$: $\forall t \in \mathbb{I}, \;\; x(t) \in K$. Then*

$$\forall t \in \mathbb{I}, \;\; x'(t) \in T_K(x(t))$$

This simple observation suggests us that the concept of tangent cone plays a crucial role for characterizing viability and invariance properties when the state space $X := \mathbb{R}^d$ is a finite-dimensional space.

Tangent cones are not necessarily convex. However, we point out that $T_K(x)$ is convex when K is convex[1]. As it happens for characterizing viability and invariance, we shall be able to replace the tangent cones by their closed convex hull:

We also need the following concept:

Definition 5.7.3. [**Convex Hulls**] *Let $K \subset X$ be a nonempty subset. We denote by $co(K)$ the smallest convex subset containing K, called the* convex hull *of K, and by $\overline{co}(K)$ the smallest closed convex subset containing K, called the* closed convex hull *of K.*

We provide now a fundamental result and its proof by Hélène Frankowska:

Theorem 5.7.4. [**Viability Tangential Conditions**] *Let $K \subset X$ be a nonempty closed subset of a finite dimensional vector space. Let x_0 belong to K. Assume that the set-valued map $F : K \rightsquigarrow X$ is upper semicontinuous with convex compact values. Then the two following properties are equivalent:*

$$\begin{cases} (i) \quad \forall x \in K \cap \overset{\circ}{B}(x_0, \alpha), \; F(x) \cap T_K(x) \; \neq \; \emptyset \\ \\ (ii) \quad \forall x \in K \cap \overset{\circ}{B}(x_0, \alpha), \; F(x) \cap \overline{co}\,(T_K(x)) \; \neq \; \emptyset \end{cases} \tag{5.16}$$

Furthermore, assume that K is compact and that

$$\forall x \in K, \; F(x) \cap \overline{co}\,(T_K(x)) \; \neq \; \emptyset$$

[1] or, more generally, when K is sleek (i.e., the tangent cone map $T_K(\cdot)$ is lower semicontinuous.)

Then, for all $\varepsilon > 0$, we can "graphically approximate" F by a set-valued map F_ε (in the sense that $\text{Graph}(F_\varepsilon) \subset \text{Graph}(F) + \varepsilon(B \times B)$*) such that*

$$\exists\, T_\varepsilon > 0 \text{ such that } \forall x \in K,\ \forall h \in [0, T_\varepsilon],\ (x + hF_\varepsilon(x)) \cap K \neq \emptyset \tag{5.17}$$

Remark: — Property (5.17) means that the discrete explicit schemes $x \rightsquigarrow x + hF_\varepsilon(x)$ associated with the graphical approximation F_ε of F are viable in K uniformly with respect to the discretization step h on compact sets. It is very useful for approximating solutions of differential inclusion $x'(t) \in F(x)$ viable in K. ■

Proof — Fix any pair of elements $x,\ y \in K$. Let us set

$$\varphi_{(x,y)}(t) := d(x + tF(y), K) = \|x + tv_t - x_t\|$$

where $v_t \in F(y)$, $x_t \in \Pi_K(x + tv_t)$, so that

$$\|x - x_t\| \leq \|x + tv_t - x_t\| + t\|v_t\| \leq \|x + tv_t - x\| + t\|v_t\| = 2t\|v_t\| \leq 2t\|F(y)\| \tag{5.18}$$

Observe that for any $u \in F(y)$, $tv_t + hu \in tF(y) + hF(y) \subset (t+h)F(y)$ because $F(y)$ is convex. Recall that for all $w \in T_K(x_t)$, there exists a sequence $e(h_n)$ converging to 0 such that $x + h_n w + h_n e(h_n) \in K$. Therefore

$$\frac{\varphi_{(x,y)}(t + h_n) - \varphi_{(x,y)}(t)}{h_n} \leq \frac{\|x + tv_t + h_n u - x_t - h_n w - h_n e(h_n)\| - \|x + tv_t - x_t\|}{h_n}$$

Dividing by h_n and passing to the limit, we infer that for all $u \in F(y)$ and $w \in \overline{\text{co}}(T_K(x_t))$

$$D_\uparrow \varphi_{(x,y)}(t)(1) \leq \left\langle u - w, \frac{x + tv_t - x_t}{\|x + tv - x_t\|} \right\rangle \leq \|u - w\|$$

and consequently, that

$$D_\uparrow \varphi_{(x,y)}(t)(1) \ \leq\ d(F(y), \overline{\text{co}}(T_K(x_t))) \tag{5.19}$$

Furthermore, F being upper semicontinuous, we can associate with any $\varepsilon > 0$ an $\eta(\varepsilon, y) \leq \varepsilon$ such that

$$\forall z \in B(y, \eta(\varepsilon, y)),\ F(z) \ \subset\ F(y) + \varepsilon B$$

1. Since (5.16)(i) implies (5.16)(ii), assume that (5.16)(ii) holds true. By taking $y = x \in \overset{\circ}{B}(x_0, \alpha)$, we deduce from (5.18) that whenever $t \leq \dfrac{\min(\eta(\varepsilon, x), \alpha - \|x_0 - x\|)}{2\|F(x)\|}$, then

$$\|x - x_t\| < \min(\eta(\varepsilon, x)), \alpha - \|x_0 - x\|) \ \&\ \|x_0 - x_t\| < \alpha$$

Assumption (5.16)(ii) implies that there exists $w_t \in F(x_t) \cap \overline{\mathrm{co}}(T_K(x_t))$ and the upper semicontinuity of F implies that there exists $u_t \in F(y)$ such that $\|u_t - w_t\| \leq \varepsilon$. Therefore

$$\begin{array}{rcl} \forall t \in \left[0, \frac{\min(\eta(\varepsilon,x)),\alpha - \|x_0 - x\|)}{2\|F(x)\|}\right],\ D_\uparrow\varphi_{(x,x)}(t)(1) & \leq & d(F(x), \overline{\mathrm{co}}(T_K(x_t))) \\ & \leq & \|u_t - w_t\| \leq \varepsilon \end{array}$$

The function $\varphi_{(x,y)}$ being lower semicontinuous, we deduce that

$$\forall t \in \left[0, \frac{\min(\eta(\varepsilon, x), \alpha - \|x_0 - x\|)}{2\|F(x)\|}\right],\ \frac{d\,(x + tF(x), K)}{t} = \left\|v_t - \frac{x_t - x}{t}\right\| \leq \varepsilon$$

Since $\|v_t\| \leq \|F(x)\|$, a subsequence $v_{t_n} := \dfrac{x_{t_n} - x}{t_n}$ converges to some $v \in F(x)$, so that the subsequence $v_{t_n} \in \dfrac{K - x}{t_n}$ converges to v. Consequently, v belongs also to the contingent cone $T_K(x)$.

2. Assume that K is compact and that

$$\forall x \in K,\ F(x) \cap \overline{\mathrm{co}}\,(T_K(x)) \neq \emptyset$$

Then there exists $w_t \in F(x_t) \cap \overline{\mathrm{co}}(T_K(x_t))$. Property (5.18) implies that $\|x - x_t\| \leq 2t\|F(y)\| \leq \dfrac{\eta(\varepsilon, y)}{2}$ whenever $t \leq \dfrac{\eta(\varepsilon, x)}{4\|F(y)\|}$, so that $\|y - x_t\| \leq \|x - y\| + \|x - x_t\| \leq \eta(\varepsilon, y)$ whenever $\|x - y\| \leq \dfrac{\eta(\varepsilon, y)}{2}$. We deduce that there exists $u_t \in F(y)$ such that $\|u_t - w_t\| \leq \varepsilon$. Therefore

$$\begin{cases} \forall t \in \left[0, \frac{\eta(\varepsilon,y)}{4\|F(y)\|}\right],\ \forall x \in K \cap B\left(y, \frac{\eta(\varepsilon,y)}{2}\right), \\ D_\uparrow\varphi_{(x,y)}(t)(1) \leq d(F(y), \overline{\mathrm{co}}(T_K(x_t))) \leq \|u_t - w_t\| \leq \varepsilon \end{cases}$$

Since $\varphi_{(x,y)}$ is lower semicontinuous, we deduce that

$$\forall t \in \left[0, \frac{\eta(\varepsilon, y)}{4\|F(y)\|}\right], \ \forall x \in K \cap B\left(y, \frac{\eta(\varepsilon, y)}{2}\right), \ d\left(x + tF(y), K\right) = \varphi_{(x,y)}(t) \leq \varepsilon t$$

The subset K being compact, it can be covered by a finite number of balls $B\left(y_j, \frac{\eta(\varepsilon, y_j)}{2}\right)$. Setting $T(\varepsilon) := \min_j \frac{\eta(\varepsilon, y_j)}{4\|F(y_j)\|} > 0$, we infer that

$$\begin{cases} \forall \varepsilon > 0, \ \exists \, T_\varepsilon > 0 \ \text{ such that } \forall x \in K, \ \exists \, y_j \in B\left(x, \frac{\eta(\varepsilon, y_j)}{2}\right) \ \text{ such that} \\ \forall t \leq T(\varepsilon), \ d\left(x + tF(y_j), K\right) = \varphi_{(x,y)}(t) \leq \varepsilon t \end{cases}$$

This means that there exist some $v_j \in F(y_j)$ and $z_j \in K$ such that $\|z_j - x - tv_j\| \leq \varepsilon t$. On the other hand,

$$\left(x, \frac{z_j - x}{t}\right) = (y_j, v_j) + \left(x - y_j, \frac{z_j - x}{t} - v_j\right) \in \text{Graph}(F) + \varepsilon(B \times B)$$

Consequently, defining the set-valued map F_ε by $\text{Graph}(F_\varepsilon) := \text{Graph}(F) + \varepsilon(B \times B)$, we have proved (5.17):

$$\exists \, T_\varepsilon > 0 \ \text{ such that } \forall x \in K, \ \forall h \in [0, T_\varepsilon], \ (x + hF_\varepsilon(x)) \cap K \ \neq \ \emptyset$$

This completes the proof. ■

Theorem 5.7.5. **[Invariance Tangential Conditions]** *Let $K \subset X$ be a nonempty closed subset of a finite dimensional vector space. Assume that the set-valued map $F : K \rightsquigarrow X$ is Lipschitz on K.*

Let x_0 belong to K and $\alpha > 0$. Then the two following properties are equivalent:

$$\begin{cases} (i) \quad \forall x \in K \cap \overset{\circ}{B}(x_0, \alpha), \ F(x) \ \subset \ T_K(x) \\ \\ (ii) \quad \forall x \in K \cap \overset{\circ}{B}(x_0, \alpha), \ F(x) \ \subset \ \overline{co}\,(T_K(x)) \end{cases} \tag{5.20}$$

Furthermore, assume that K is compact and that

$$\forall x \in K,\ F(x) \subset \overline{co}\,(T_K(x)) \neq \emptyset$$

Then, for all $\varepsilon > 0$,

$$\exists\, T_\varepsilon > 0 \ \text{ such that} \, \forall x \in K,\ \forall h \in [0, T_\varepsilon]\,,\ x + hF(x) \subset K + \varepsilon h B \neq \emptyset \qquad (5.21)$$

Remark: — Property (5.21) means that the discrete explicit schemes $x \rightsquigarrow x + hF(x)$ associated with F are invariant in approximations $K + \varepsilon h B$ uniformly with respect to the discretization step h on compact sets. ■

Proof — Assume that (5.20)(ii) holds true. We associate with any $v \in F(x)$ the function

$$\varphi_{(x,v)}(t) := d(x + tv, K) = \|x + tv - x_t\|$$

where $x_t \in \Pi_K(x + tv)$. Recall that for all $w \in T_K(x_t)$, there exists a sequence $e(h_n)$ converging to 0 such that such that $x + h_n w + h_n e(h_n) \in K$. Therefore

$$\frac{\varphi_{(x,v)}(t + h_n) - \varphi_{(x,v)}(t)}{h_n} \leq \frac{\|x + tv + h_n(v - w) - x_t - h_n e(h_n)\| - \|x + tv - x_t\|}{h_n}$$

Dividing by h_n and passing to the limit, we infer that

$$D_\uparrow \varphi_{(x,v)}(t)(1) \leq \left\langle v - w, \frac{x + tv - x_t}{\|x + tv - x_t\|} \right\rangle$$

and thus, that this inequality holds true for all $w \in \overline{\mathrm{co}}(T_K(x_t))$.

Furthermore, we can associate with any $v \in F(x)$ an element $u_t \in F(x + tv)$ such that $\|v - u_t\| = d(v, F(x + tv))$ and, F being Lipschitz, we associate with $u_t \in F(x + tv)$ an element $w_t \in F(x_t) \subset \overline{\mathrm{co}}(T_K(x_t))$ such that

$$\left\langle u_t - w_t, \frac{x + tv - x_t}{\|x + tv - x_t\|} \right\rangle \ \leq\ \lambda \|x + tv - x_t\| = \lambda \varphi_{(x,v)}(t)$$

Therefore,

$$D_\uparrow \varphi_{(x,v)}(t)(1) \leq \left\langle v - w, \frac{x + tv - x_t}{\|x + tv - x_t\|} \right\rangle \leq \lambda \varphi_{(x,v)}(t) + d(v, F(x + tv))$$

Since $\varphi_{(x,v)}$ is lower semicontinuous, we deduce that

$$\forall t \geq 0,\ \varphi_{(x,v)}(t) \leq e^{\lambda t}\left(\varphi_{(x,v)}(0) + \int_0^t e^{-\lambda s} \sup_{v\in F(x)} d(v, F(x+sv))ds\right)$$

Therefore, for every $x \in K$, $\varphi_{(x,v)}(0) = 0$, so that

$$\forall v \in F(x),\ \forall t \geq 0,\ \left\| v - \frac{x_t - x}{t} \right\| \leq \sup_{s\in[0,t]} \sup_{v\in F(x)} d(v, F(x+sv)) \frac{e^{\lambda t} - 1}{\lambda t}$$

Since F is Lipschitz, property

$$\begin{cases} \forall (x_0, v_0) \in \mathrm{Graph}(F),\ \forall \varepsilon > 0,\ \exists\, T_{(\varepsilon,x_0,v_0)} > 0 \text{ such that} \\ \qquad \sup_{s\in[0,T_{(\varepsilon,x_0,v_0)}]} d(v_0, F(x_0 + sv_0)) \leq \varepsilon \end{cases} \tag{5.22}$$

holds true. Therefore

$$\forall t \in \left[0, T_{(\varepsilon,x,v)}\right],\ \left\| v - \frac{x_t - x}{t} \right\| \leq \varepsilon \max\left(1, \frac{e^{\lambda T_{(\varepsilon,x)}} - 1}{\lambda T_{(\varepsilon,x)}}\right)$$

We infer that for any $\varepsilon > 0$, there exists $T_{(\varepsilon,x,v)} > 0$ such that

$$\forall t \in \left[0, T_{(\varepsilon,x)}\right],\ x + tF(x)\ \subset\ K + t\varepsilon B$$

Furthermore, $\left\| v - \frac{x_t - x}{t} \right\|$ converges to 0 with t. Since $x_t \in K$, we infer that v belongs also to the contingent cone $T_K(x)$.

Actually, since F is Lipschitz, for any $\varepsilon > 0$, there exist $T_{(\varepsilon,x)} > 0$ and $\alpha_{(\varepsilon,x)} > 0$ such that

$$\sup_{s\in[0,T_{(\varepsilon,x)}]} \sup_{y\in B(x,\alpha_{(\varepsilon,x)})\cap K} \sup_{v\in F(x)} d(v, F(x+sv)) \leq \varepsilon \tag{5.23}$$

holds true. Then we deduce that

$$\forall t \in \left[0, T_{(\varepsilon,x)}\right], \forall y \in B(x, \alpha_{(\varepsilon,x)}) \cap K,\ y + tF(y)\ \subset\ K + t\varepsilon B$$

and that $F(x) \subset C_K(x)$ is contained in the Clarke tangent cone.

Consequently, if K is compact, we infer that for any $\varepsilon > 0$, there exists T_ε such that

$$\forall h \in [0, T_\varepsilon],\ \forall x \in K,\ x + hF(x)\ \subset\ K + \varepsilon h B$$

by covering K by a finite number of balls $B(x_j, \alpha_{(\varepsilon,x_j)})$ and by taking $T_\varepsilon := \min_j T_{(\varepsilon,x)_j}$. ■

Remark: — Observe that the function e_λ defined by

$$e_\lambda(t) := \frac{e^{\lambda t} - 1}{\lambda t} \text{ if } \lambda \neq 0 \text{ and } e_0(t) := t$$

satisfies $e_\lambda(0) = 1$, $e'_\lambda(0) = \frac{\lambda}{2}$, is decreasing and converges to 0 when $t \to +\infty$ if $\lambda < 0$ and is increasing and converges to $+\infty$ when $t \to +\infty$ if $\lambda > 0$. ■

5.8 Viability Theorems

5.8.1 The Basic Viability Theorem

All results on properties of evolutions governed by differential inclusions, such as local viability and capturability, hold true for the class of Marchaud set-valued maps.

We state without proof the main Viability Theorem:

Theorem 5.8.1. [**The Basic Viability Theorem**] *Let $K \subset X$ and $C \subset K$ be two closed subsets. Assume that F is Marchaud. Then the two following statements are equivalent*

1. *K is viable outside C under F,*
2. *The tangential condition*

$$\forall x \in K \backslash C,\ F(x) \cap \overline{co}(T_K(x)) \neq \emptyset \tag{5.24}$$

   holds true.

In particular, when the target C is empty, K is viable under F if and only if the above tangential condition (5.24) holds true for any $x \in K$.

The Viability Theorem has a long history. It began in the case of differential equations in 1942 with the Japanese mathematician Nagumo in a paper written in German (who however did not relate its tangential condition to the Bouligand-Severi tangent cone). The Nagumo Theorem has been rediscovered many times since.

The Viability Theorem for differential inclusions has been proved independently at the end of the 1970's by Bebernes and Shur, Gautier, Haddad (in the case of functional differential inclusions) and Yorke.

5.8.2 Implicit Differential inclusions

Theorem 5.8.2. *Assume that a set-valued map map $\Phi : X \times X \rightsquigarrow Y$ has a closed graph and that there exists a constant $c > 0$ that for every $x \in X$, $d(0, \Phi(x, v)) \leq c(\|x\| + 1)$, that $v \mapsto \Phi(x, v)$ is a convex process, and that there exists $v \in T_K(x) \cap c(\|x\| + 1)B$ such that $0 \in \Phi(x, v)$. Then, for any $x_0 \in K$, there exists a solution $x(\cdot)$ to implicit differential inclusion $0 \in \Phi(x(t), x'(t))$ satisfying $x(0) = x_0$ and viable in K.*

Proof — We associate with Φ the set-valued map $F : X \rightsquigarrow X$ defined by

$$F(x) := \{v \text{ such that } d(0, \Phi(x, v)) = 0\}$$

By assumption, $\|F(x)\| \leq c(\|x\| + 1)$ and $F(x) \in T_K(x) \neq \emptyset$. The images are convex because the function $v \mapsto d(0, \Phi(x, v))$ is convex whenever $v \rightsquigarrow \Phi(x, v)$ is a convex process: inclusion $\sum_i \alpha_i \Phi(x, v_i) \subset \Phi(x, \sum_i \alpha_i v_i)$. The graph of F is closed: Let $(x_n, u_n) \in \text{Graph}(F)$ converge to (x, u). Let $z_n \in \Phi(x_n, u_n)$ such that $\|z_n\| = d(0, \Phi(x_n, u_n)) \leq c(\|x_n\| + 1) \leq c(\|x\| + 2)$ for n large enough. A subsequence (again denoted by) z_n converges to some $z \in \Phi(x, u)$ (since the graph of Φ is assumed to be closed) satisfying $d(0, \Phi(x, u)) \leq \|z\|$. Consequently, inequalities $-c(\|x_n\| + 1) \leq -\|z_n\| = -d(0, \Phi(x_n, u_n))$ imply by passing to the limit inequalities $-c(\|x\| + 1) \leq -\|z\| \leq -d(0, \Phi(x, u))$. Hence $(x, u, \eta) \in \text{Graph}(F)$.

Therefore, the Viability Theorem implies that from $x \in K$ starts a solution $t \mapsto (x(t)$ to the differential inclusion $x'(t) \in F(x(t))$ which can be written $d(0, \Phi(x(t), x'(t))) = 0$, i.e., $0 \in \Phi(x(t), x'(t))$. ■

5.8.3 Filippov Maps

Results dealing with tychastic properties under differential inclusions, such as local invariance and absorption, hold true for the class of Filippov set-valued maps:

The celebrated Filippov Theorem states that a Lipschitz set-valued map with closed values satisfies the Filippov property:

Theorem 5.8.3. Filippov Maps. *Let $\lambda \in \mathbb{R}$ be any real number (positive, negative* *or nul). A set-valued map $F : X \rightsquigarrow X$ is called a* λ-Filippov map *if for any $\xi(\cdot)$: $W^{1,\infty}(0,\infty;X)$ such that $t \to d(\xi'(t), F(\xi(t)))$ is integrable for the measure $e^{-\lambda s}ds$, there exists a solution $x(\cdot) \in \mathcal{S}_F(x)$ to differential inclusion $x' \in F(x)$ such that, for all $t \geq 0$, the Filippov inequality*

$$\forall t \geq 0,\ \|x(t) - \xi(t)\| \leq e^{\lambda t}\left(\|x - \xi(0)\| + \int_0^t d(\xi'(s), F(\xi(s)))e^{-\lambda s}ds\right) \tag{5.25}$$

holds true. The evolutionary system $\mathcal{S}_F : X \rightsquigarrow \mathcal{C}(0\infty,;X)$ associated with a Filippov set-valued is thus called a **Filippov evolutionary system**.

We deduce at once from the definition of Filippov maps the following consequence:

Theorem 5.8.4. Lower Semicontinuity of Filippov Evolutionary Systems. *Assume that a strict set-valued map $F : X \rightsquigarrow X$ is λ-Filippov. Then the associated evolutionary system $\mathcal{S}_F$ is lower semicontinuous (if $x_n \in X$ converge to x in X and $x(\cdot) \in \mathcal{S}_F(x)$, then there exist solutions $x_n(\cdot) \in \mathcal{S}_F(x_n)$ to the differential inclusion $x' \in F(x)$ starting at x_n converging to a solution $x(\cdot) \in \mathcal{S}_F(x)$ uniformly on compact intervals).*

Theorem 5.8.5. Filippov Theorem. *Lipschitz maps with closed values are Filippov maps.*

Proof — We do not provide the proof of the Filippov Theorem. ■

5.8.4 Tangential Characterization of Invariance

We provide sufficient and necessary conditions for invariance couched in terms of tangential conditions.

Proposition 5.8.6. Sufficient Conditions for Invariance. *Assume that F is Lipschitz. Then condition*

$$\forall x \in K\backslash C,\ F(x)\ \subset\ \overline{co}(T_K(x))$$

implies that K is invariant outside C under F.

Proof Let us assume that $F(y) \subset \overline{\text{co}}\,(T_K(y))$ on a neighborhood of $x \in K \setminus C$ and let $x(\cdot) \in \mathcal{S}(x_0)$ be any solution to differential inclusion $x' \in F(x)$ starting at x_0 in a neighborhood x and defined on some interval $[0,T]$. Let t be a point such that both $x'(t)$ exists and $x'(t)$ belongs to $F(x(t))$. Then there exists $\varepsilon(h)$ converging to 0 with h such that

$$x(t+h) = x(t) + hx'(t) + h\varepsilon(h)$$

Introduce

$$\varphi(t) := d(x(t), K) = \|x(t) - x_t\|$$

where $x_t \in \Pi_K(x(t))$. Recall that for all $w \in T_K(x_t)$, there exists a sequence $e(h_n)$ converging to 0 such that such that $x + h_n w + h_n e(h_n) \in K$. Therefore

$$\frac{\varphi(t+h_n) - \varphi(t)}{h_n} \leq \frac{\|x(t) + h_n(x'(t) - w) - x_t - h_n e(h_n)\| - \|x(t) - x_t\|}{h_n}$$

Dividing by h_n and passing to the limit, we infer that

$$D_\uparrow \varphi(t)(1) \leq \left\langle x'(t) - w, \frac{x(t) - x_t}{\|x(t) - x_t\|} \right\rangle$$

and thus, that this inequality holds true for all $w \in \overline{\text{co}}(T_K(x_t))$.

Since F is Lipschitz, there exists a constant $\lambda \in \mathbf{R}$ and $w_t \in F(x_t) \subset \overline{\text{co}}(T_K(x_t))$ such that

$$\langle x(t) - x_t, x'(t) - w_t \rangle \leq \lambda \|x(t) - x_t\|^2$$

Therefore

$$D_\uparrow \varphi(t)(1) \leq \left\langle x'(t) - w_t, \frac{x(t) - x_t}{\|x(t) - x_t\|} \right\rangle \leq \lambda \|x(t) - y\| = \lambda \varphi(t)$$

Then φ is a lower semicontinuous solution to $D_\uparrow \varphi(t)(1) \leq \lambda \varphi(t)$ and thus satisfies inequality $\varphi(t) \leq \varphi(0) e^{\lambda t}$. Since $\varphi(0) = 0$, we deduce that $\varphi(t) = 0$ for all $t \in [0,T]$, and therefore that $x(t)$ is viable in K on $[0,T]$. ■

A necessary condition requires the existence of a solution to a differential inclusion starting with both an initial state and initial velocity:

Proposition 5.8.7. Necessary Conditions for Invariance. *Assume that for any*

$(x_0, v_0) \in \text{Graph}(F)$, *there exists a solution* $x(\cdot) \in \mathcal{S}_F(x_0)$ *satisfying* $x'(0) = v_0$. *If* K *is invariant outside* C *under* F, *then*

$$\forall x \in K \backslash C,\ F(x)\ \subset\ T_K(x)$$

Proof — Let $x_0 \in K\backslash C$. We have to prove that any $u_0 \in F(x_0)$ is tangent to K at x_0. By assumption, for all x_0 and $v_0 \in F(x_0)$, there exists a solution $x(\cdot)$ to the differential inclusion $x' \in F(x)$ satisfying $x(0) = x_0$ and $x'(0) = v_0$ viable in $K \setminus C$. Hence v_0, being the limit of $\dfrac{x(t_n) - x_0}{t_n} \in \dfrac{K - x_0}{t_n}$, it belongs to $T_K(x_0)$. It follows that $F(x_0)$ is contained in $T_K(x_0)$. ■

This motivates the study of initial state-velocity value problems.

The question arises whether there exists a solution to the strong initial value problem where both the initial state x_0 and initial velocity $v_0 \in F(x_0)$ are fixed.

Proposition 5.8.8. Initial State-Velocity Value Problem for Lipschitz Maps. *Assume that a set-valued map* $F : X \rightsquigarrow X$ *is Lipschitz. Then, for any* $x_0 \in X$ *and* $v_0 \in F(x_0)$, *there exists a solution* $x(\cdot)$ *to differential inclusion* $x' \in F(x)$ *satisfying* $x(0) = x_0$ *and* $x'(0) = v_0$.

Results dealing with tychastic properties under differential inclusions, such as local invariance and absorption, hold true for the class of Lipschitz set-valued maps.

The Invariance Theorem states that K is invariant if and only all velocities are tangent to K:

Theorem 5.8.9. The Basic Invariance Theorem. *Let* $K \subset X$ *and* $C \subset K$ *be two closed subsets. Assume that* F *is Lipschitz Then the two following statements are equivalent*

1. K *is invariant outside* C *under* F
2. *The tangential condition*

$$\forall x \in K \backslash C,\ F(x)\ \subset\ \overline{co}(T_K(x)) \tag{5.26}$$

   holds true.

In particular, when the target C is empty, K is invariant under F if and only if the above tangential condition (5.26) holds true for any $x \in K$.

The Invariance Theorem for Lipschitz maps is based on a generalization of the Cauchy-Lipschitz Theorem by Filippov in the early 1960's. The Viability Theorem for differential inclusions was proved by Frank Clarke in the 1970's.

5.9 Frankowska's and Viscosity Property of Viability Kernels

Viability kernels are now characterized in terms of tangential conditions:

Proposition 5.9.1. [**Tangential Characterization of Viability Kernels**] *Assume that F is Marchaud and that $C \subset K$ and K are closed. The viability kernel* $\mathrm{Viab}_F(K, C)$ *of the subset K with target C under F is the* **largest** *closed subset D of K satisfying*

$$\forall x \in D \backslash C,\ F(x) \cap \overline{co}(T_D(x)) \neq \emptyset$$

We shall provide tangential properties of the viability kernel in terms of tangential conditions to a viability kernel. They are at the origin of theorems stating that value functions in optimal control theory are generalized solutions to Hamilton-Jacobi partial differential equation introduced in the early 1990 independently by Emmanuel Barron and Robert Jensen with partial differential techniques and Hélène Frankowska with viability techniques, when the subsets involved are epigraphs of functions:

Definition 5.9.2. [**Property of a Set**] *Let us consider a set-valued map $F : X \rightsquigarrow X$ and two subsets $C \subset K$ and K. We shall say that a subset D between C and K satisfies the* Frankowska property *with respect to F if*

$$\left\{ \begin{array}{ll} i) & \forall x \in D \backslash C,\ F(x) \cap T_D(x) \neq \emptyset \\ ii) & \forall x \in D \cap \mathrm{Int}(K),\ -F(x) \subset T_D(x) \\ iii) & \forall x \in D \cap \partial K,\ -F(x) \subset T_D(x) \cup T_{\complement K}(x) \end{array} \right. \tag{5.27}$$

Remark: — When K is assumed further to be backward locally invariant and F to be Filippov, the above condition (5.27) and boils down to

$$\begin{cases} i) & \forall x \in D\backslash C,\ F(x) \cap T_D(x) \neq \emptyset \\ ii) & \forall x \in D,\ -F(x) \subset T_D(x) \ \blacksquare \end{cases} \tag{5.28}$$

We deduce from the Characterization Theorem 5.4.10 and the Basic Viability and Invariance Theorems the tangential characterization of viability kernels:

Theorem 5.9.3. [**The Frankowska Property of Viability Kernels**] *Let us assume that F is both Marchaud and Filippov and that $C \subset K$ and K are closed. The viability kernel* $\mathrm{Viab}_F(K, C)$ *of the subset K with target C under F is the* **unique** *closed subset $D \subset K$ satisfying*

1. *the Frankowska property (5.27),*
2. *$K\backslash D$ is a repeller.*

It may be useful to provide tangential properties of the viability kernel in terms of tangential conditions to the complement of the viability kernel. They are at the origin of theorems stating that value functions in optimal control theory are viscosity solutions to Hamilton-Jacobi partial differential equation introduced in 1983 by Michael Crandall and Pierre-Louis Lions, when the subsets involved are epigraphs of functions:

Definition 5.9.4. [**Viscosity Property**] *Let us consider a set-valued map $F : X \rightsquigarrow X$ and two subsets $C \subset K$ and K. We shall say that a subset D between C and K satisfies the* viscosity property *with respect to F if*

$$\begin{cases} i) & \forall x \in D\backslash C,\ F(x) \cap T_D(x) \neq \emptyset \\ ii) & \forall x \in \mathrm{Int}(K)\backslash\mathrm{Int}(D),\ F(x) \subset T_{\overline{\complement D}}(x) \end{cases} \tag{5.29}$$

We thus deduce from the Basic Viability and Invariance Theorems the following

Proposition 5.9.5. [**The Viscosity Property of Capture Basins**] *Let us assume that F is Marchaud and Filippov, that $C \subset K$ and K are closed and that $K\backslash C$ is a repeller. Then the viable-capture basin* $\mathrm{Capt}_F(K, C)$ *of the target C viable in K under F satisfies* *the viscosity property (5.29).*

Theorem 5.5.1 implies the tangential characterization of invariance kernels:

Invariance kernels are also characterized in terms of tangential conditions:

Proposition 5.9.6. [**Tangential Characterization of Invariance Kernels**] *Assume that F is Lipschitz and that K and $C \subset K$ are closed. Then the invariance kernel* $\mathrm{Inv}_{\mathcal{S}}(K, C)$ *of K with target C under F is the* largest closed subset D *between C and K such that*

$$\forall x \in D \backslash C, \; F(x) \; \subset \; \overline{co}(T_D(x))$$

About guaranteed viability kernels, it can be proven that

Theorem 5.9.7. *The guaranteed viable-capture basin* $[\mathrm{Capt}_P \mathrm{Abs}_V](K, C)$ *is the largest subset between C and K is the largest fixed point of the map* $D \mapsto [\mathrm{Capt}_P \mathrm{Abs}_V](D, C)$.

Consequently, the guaranteed viable-capture basin satisfies

$$[\mathrm{Capt}_P \mathrm{Abs}_V](K, C) \; = \; [\mathrm{Capt}_P \mathrm{Abs}_V]([\mathrm{Capt}_P \mathrm{Abs}_V](K, C), C)$$

In other words, it is the largest subset of elements $x \in K$ such that there exists a feedback $\widetilde{u} \in \widetilde{\mathcal{P}}$ such that for every solutions $(x(\cdot), v(\cdot)) \in \mathcal{C}_{\widetilde{u}}(x)$, there exists $t^* \in \mathbf{R}_+$ satisfying the viability/capturability conditions.

We shall assume that the dynamical game (3.20) is Lipschitz in the sense that the set-valued maps P and Q are Lipschitz with compact values and that the single-valued map f is Lipschitz with closed values.

Let $\widetilde{\mathcal{P}}_\lambda$ be the set of Lipschitz selections with constant λ of the set-valued map P: for every x, $\widetilde{u}(x) \in U(x)$.

The subset

$$[\mathrm{Capt}_{P_\lambda} \mathrm{Abs}_V](K, C) \; := \; \bigcup_{\widetilde{u} \in \widetilde{\mathcal{P}}_\lambda} \mathrm{Abs}_{\widetilde{u}}(K, C)$$

is called the λ-guaranteed viable-capture basin of a target under the evolutionary game $(x, \widetilde{u}) \rightsquigarrow \mathcal{C}_{\widetilde{u}}(x)$ associated with the dynamical game (3.20).

One can prove that when the game is Lipschitz, the set-valued map $(x, \widetilde{u}) \in X \times \widetilde{\mathcal{P}}_\lambda \rightsquigarrow \mathcal{C}_{\widetilde{u}}(x) \subset \mathcal{C}(0, \infty; X)$ is lower semicontinuous and consequently, that the λ-guaranteed viable-capture basin is closed.

Using the Viability and the Invariance Theorems, one can prove the following tangential properties of guaranteed viability kernels with targets:

Theorem 5.9.8. *Let us assume that the dynamical game (U, Q, f) is Lipschitz, that $C \subset K$ and K are closed subsets of X and that $K \backslash C$ is a repeller under all the maps $(x, \widetilde{u}) \rightsquigarrow \mathcal{C}_{\widetilde{u}}(x)$.*

Then the λ-guaranteed viable-capture basin $[\mathrm{Capt}_{P_\lambda} \mathrm{Abs}_V](K, C)$ of target C viable in K is the largest of the closed subsets D satisfying $C \subset D \subset K$ and

1. *the tangential property*[2]
$$\forall\, x \in D \backslash C,\ \exists\, u \in U(x) \ \text{ such that} \ \forall\, v \in V(x),\ \ f(x, u, v) \ \in \ T_D(x) \tag{5.30}$$
2. *there exists a λ-Lipschitz selection of the guaranteed regulation map Γ_D defined by*
$$\forall\, x \in D \backslash C,\ \ \Gamma_D(x) \ := \ \{u \in U(x) \mid f(x, u, V(x)) \subset T_D(x)\}$$

5.10 Convergence Theorems

5.10.1 Finite-Difference Approximations

Definition 5.10.1. *Let us consider a Marchaud map $F : X \rightsquigarrow X$. Discretizations of F are set-valued maps $G_\rho : X \rightsquigarrow X$ satisfying*
$$\forall \varepsilon > 0,\ \exists\, \rho_\varepsilon > 0 \mid \forall \rho \in]0, \rho_\varepsilon],\ \mathrm{Graph}\left(\frac{G_\rho - \mathbf{1}}{\rho}\right) \ \subset \ \mathrm{Graph}(F) + \varepsilon B$$

This is naturally the case if we take the explicit discretization $G_\rho^\alpha := \mathbf{1} + \rho F + \alpha \rho^2 B$ for some $\alpha \geq 0$.

We associate with any solution $\vec{x}^\rho := (x_0^\rho, \ldots, x_n^\rho, \ldots)$ to the discrete system G_ρ the piecewise linear function $\mathbf{p}_\rho \vec{x}^\rho \in \mathcal{C}(0, \infty; X)$ interpolating this sequence at the nodes $n\rho$:
$$\forall n \geq 0,\ \forall t \in [n\rho, (n+1)\rho[,\ \mathbf{p}_\rho \vec{x}^\rho(t) := x_n^\rho + \frac{x_{n+1}^\rho - x_n^\rho}{\rho}(t - n\rho)$$

[2] or, the equivalent dual formulation,
$$\forall\, x \in D \backslash C,\ \forall\, p \in N_D(x),\ \ \inf_{u \in U(x)} \sup_{v \in V(x)} \langle f(x, u, v), p \rangle \ \leq \ 0$$
where the (regular) normal cone $N_D(x) := T_D(x)^-$ is the polar cone to the contingent cone $T_D(x)$.

On the other hand, we denote by $\mathbf{r}_\rho$ the map which associates with any continuous function $x(\cdot) \in \mathcal{C}(0, \infty; X)$ the sequence $\mathbf{r}_\rho$ defined by

$$\forall j \geq 0,\ \mathbf{r}_\rho x^j := x(\rho j)$$

We observe that any continuous function $x(\cdot)$ can be approximated by the functions $\mathbf{p}_\rho \mathbf{r}_\rho x(\cdot)$.

Theorem 5.10.2. *Let us consider a sequence of discretizations G_ρ of a Marchaud map $F : X \rightsquigarrow X$ and a sequence of solutions $\vec{x}^\rho$ to the discrete dynamical system*

$$\forall n \geq 0,\ x_{n+1}^\rho \in G_\rho(x_n^\rho)$$

Then there exists a subsequence (again denoted by) $\vec{x}^\rho$ such that $\mathbf{p}_\rho \vec{x}^\rho$ converges uniformly on compact intervals to a solution to the differential inclusion $x' \in F(x)$.

Proof — Let us consider solutions $\vec{x}^\rho := (x_0^\rho, \ldots, x_n^\rho, \ldots)$ to the discrete system G_ρ. Then the functions $x_\rho := \mathbf{p}_\rho \vec{x}^\rho$ satisfy for almost all $t \geq 0$

$$(x_\rho(t), x_\rho'(t)) \in \text{Graph}(F) + \varepsilon B$$

Convergence Theorem 5.10.3 implies that this limit $x(\cdot)$ is a solution to the differential inclusion $x' \in F(x)$. □

Theorem 5.10.3. *Let $F : X \rightsquigarrow X$ be a Marchaud map. Consider a sequence of m-approximate solutions $x_m(\cdot)$ in the sense that*

$$\forall t \in [0, T],\ d((x_m(t), x_m'(t)), \text{Graph}(F) \leq \frac{1}{m} \tag{5.31}$$

such that $x_m(0)$ converges to x_0. Then a subsequence (again denoted by) $x_m(\cdot)$ converges uniformly on compact intervals to a solution $x(\cdot)$ to the differential inclusion $x' \in F(x)$.

Proof — Consider a sequence of n-approximate solutions $x_m(\cdot)$. They satisfy the following a priori estimates:

$$\|x_m(t)\| \leq \left(\|x_0\| + 1 + \frac{c+1}{mc}\right) e^{ct} \ \&\ \|x_m'(t)\| \leq c\left(\|x_0\| + 1 + \frac{c+1}{mc}\right) e^{ct} \tag{5.32}$$

Indeed, the function $t \to \|x_m(t)\|$ being locally Lipschitz, it is almost everywhere differentiable. Therefore, for any t where $x_m(t)$ is different from 0 and differentiable, we have

$$\frac{d}{dt}\|x_m(t)\| = \left\langle \frac{x_m(t)}{\|x_m(t)\|}, x'_m(t) \right\rangle \leq \|x'_m(t)\|$$

Since there exist elements $u_t \in \frac{1}{m}B_X$ and $v_t \in \frac{1}{m}B_X$ such that

$$x'_m(t) \in F(x_m(t) + u_t) + v_t$$

we obtain

$$\|x'_m(t)\| \leq c(\|x_m(t)\| + 1 + \frac{1}{m}) + \frac{1}{m}$$

Setting $\varphi(t) := \|x_m(t)\| + 1 + \dfrac{1}{m}\dfrac{c+1}{c}$, we infer that $\varphi'(t) \leq c\varphi(t)$, and thus

$$\varphi(t) \leq \varphi(0)e^{ct}$$

from which we deduce the estimates (5.32).

Estimates (5.32) imply that for all $t \in [0, T]$, the sequence $x_m(t)$ remains in a bounded set and that the sequence $x_m(\cdot)$ is equicontinuous, because the derivatives $x'_m(\cdot)$ are bounded. We then deduce from Ascoli's Theorem that it remains in a compact subset of the Banach space $\mathcal{C}(0, T; X)$, and thus, that a subsequence (again denoted) $x_m(\cdot)$ converges uniformly to some function $x(\cdot)$.

Furthermore, the sequence $x'_m(\cdot)$ being bounded in the dual of the Banach space $L^1(0, T; X)$, which is equal to $L^\infty(0, T; X)$, it is weakly relatively compact thanks to Alaoglu's Theorem[3]. The Banach space $L^\infty(0, T; X)$ is contained in $L^1(0, T; X)$ with a stronger topology[4]. The identity map being continuous for the norm topologies, is still continuous for the weak topologies. Hence the sequence $x'_m(\cdot)$ is weakly relatively compact in $L^1(0, T; X)$ and a subsequence (again denoted) $x'_m(\cdot)$ converges weakly to some function $v(\cdot)$ belonging to $L^1(0, T; X)$. Equations

$$x_m(t) - x_m(s) \;=\; \int_s^t x'_m(\tau)d\tau$$

[3] Alaoglu's Theorem states that any bounded subset of the dual of a Banach space is weakly compact.

[4] Since the Lebesgue measure on $[0, T]$ is finite, we know that

$$L^\infty(0, T; X) \subset L^1(0, T; X)$$

with a stronger topology. The weak topology $\sigma(L^\infty(0, T; X),\ L^1(0, T; X))$ (weak-star topology) is stronger than the weakened topology $\sigma(L^1(0, T; X),\ L^\infty(0, T; X))$ since the canonical injection is continuous. Indeed, we observe that the seminorms of the weakened topology on $L^1(0, T; X)$, defined by finite sets of functions of $L^\infty(0, T; X)$, are seminorms for the weak-star topology on $L^\infty(0, T; X)$), since they are defined by finite sets of functions of $L^1(0, T; X)$.

imply that this limit $v(\cdot)$ is actually the weak derivative $x'(\cdot)$ of the limit $x(\cdot)$.

In summary, we have proved that

$$\begin{cases} i) \quad x_m(\cdot) \text{ converges uniformly to } x(\cdot) \\ \\ ii) \quad x'_m(t) \text{ converges weakly to } \ x'(\cdot) \text{ in } L^1(0,T;X) \end{cases}$$

Let us recall that in a Banach space, the closure (for the normed topology) of a set coincides with its weak closure (for the weakened topology[5]).

We apply this result: for every m, the function $y(\cdot)$ belongs to the weak closure of the convex hull $\text{co}(\{y_p(\cdot)\}_{p\geq m}.)$ It coincides with the (strong) closure of $\text{co}(\{y_p(\cdot)\}_{p\geq m})$. Hence we can choose functions

$$v_m(\cdot) := \sum_{p=m}^{\infty} a_m^p y_p(\cdot) \in \text{co}(\{y_p(\cdot)\}_{p\geq m})$$

(where the coefficients a_m^p are positive or equal to 0 but for a finite number of them, and where $\sum_{p=m}^{\infty} a_m^p = 1$) which converge strongly to $y(\cdot)$ in $L^1(0,T;X)$. This implies that the sequence $v_m(\cdot)$ converges strongly to the function $y(\cdot)$ in $L^1(0,T;X)$.

Thus, there exists another subsequence (again denoted by) $v_m(\cdot)$ such that[6]

$$\text{for almost all } \ t \in [0,T], \ \ v_m(t) \ \text{ converges to } y(t)$$

[5] By definition of the weakened topology, the continuous linear functionals and the weakly continuous linear functionals coincide. Therefore, the closed half-spaces and weakly closed half-spaces are the same. The Hahn-Banach Separation Theorem, which holds true in Hausdorff locally convex topological vector spaces, states that closed convex subsets are the intersection of the closed half-spaces containing them. Since the weakened topology is locally convex, we then deduce that closed convex subsets and weakly closed convex subsets do coincide. This result is known as Mazur's theorem.

[6] Strong convergence of a sequence in Lebesgue spaces L^p implies that some subsequence converges almost everywhere. Let us consider indeed a sequence of functions f_n converging strongly to a function f in L^p. We can associate with it a subsequence f_{n_k} satisfying

$$\|f_{n_k} - f\|_{L^p} \leq 2^{-k}; \ \cdots < n_k < n_{k+1} < \cdots$$

Therefore, the series of integrals

$$\sum_{k=1}^{\infty} \int \|f_{n_k}(t) - f(t)\|_X^p dt < +\infty$$

is convergent. The Monotone Convergence Theorem implies that the series

$$\sum_{k=1}^{\infty} \|f_{n_k}(t) - f(t)\|_X^p$$

converges almost everywhere. For every t where this series converges, we infer that the general term converges to 0.

— Let $t \in [0,T]$ such that $x_m(t)$ converges to $x(t)$ in X and $v_m(t)$ converges to $y(t)$ in X. Let $p \in X^\star$ be such that $\sigma(F(x(t)),p) < +\infty$ and let us choose $\lambda > \sigma(F(x(t)),p)$. Since F is upper hemicontinuous, there exists a neighborhood $\mathcal{V}$ of 0 in X such that

$$\forall u \in x(t) + \mathcal{V}, \quad \text{then } \sigma(F(u),p) \leq \lambda \tag{5.33}$$

Let N_1 be an integer such that

$$\forall q \geq N_1, \quad x_q \in x(t) + \frac{1}{2}\mathcal{V}$$

Let $\eta > 0$ be given. Assumption (5.31) of the theorem implies the existence of N_2 and of elements (u_q, v_q) of the graph of F such that

$$\forall q \geq N_2, \ u_q \in x_q(t) + \frac{1}{2}\mathcal{V}, \ \ \|y_q(t) - v_q\| \leq \eta$$

Therefore u_q belongs to $x(t) + \mathcal{V}$ and we deduce from (5.33) that

$$\begin{cases} < p, y_q(t) > & \leq & < p, v_q > + \eta\|p\|_\star \\ & \leq & \sigma(F(u_q),p) + \eta\|p\|_\star \\ & \leq & \lambda + \eta\|p\|_\star \end{cases}$$

Let us fix $N \geq \max(N_1, N_2)$, multiply the above inequalities by the nonnegative a_m^q and add them up from $q = 1$ to ∞. We obtain :

$$< p, v_m(t) > \leq \lambda + \eta\|p\|_\star$$

By letting m go to infinity, it follows that

$$< p, y(t) > \leq \lambda + \eta\|p\|_\star$$

Letting now λ converge to $\sigma(F(x(t)),p)$ and η to 0, we obtain:

$$< p, y(t) > \leq \sigma(F(x(t)),p)$$

Since this inequality is automatically satisfied for those p such that

$$\sigma(F(x(t)),p) = +\infty$$

it thus holds true for every $p \in X^\star$. Hence, the images $F(x)$ being closed and convex, the Separation Theorem implies that $y(t)$ belongs to $F(x(t))$. The Convergence Theorem ensues. □

Theorem 5.10.4. *Let us assume that $F : X \rightsquigarrow X$ is Lipschitz and bounded by $\|F\|_\infty := \sup_{x\in X} \|F(x)\| < +\infty$. We set*

$$\alpha := \frac{\|F\|_\Lambda \|F\|_\infty}{2}$$

Setting

$$G_\rho^\alpha := \mathbf{1} + \rho F + \alpha\rho^2 B$$

the map $\mathbf{r}_\rho$ sends the solution map $\mathcal{S}_F$ into the solution map $\vec{\mathcal{S}}_{G_\rho^\alpha}$:

$$\mathbf{r}_\rho\left(\mathcal{S}_F(x_0)\right) \ \subset \ \vec{\mathcal{S}}_{G_\rho^\alpha}(x_0) \tag{5.34}$$

For any $\beta \geq 0$, the map $\mathbf{p}_\rho$ satisfies

$$\left\{ \begin{array}{l} \mathbf{p}_\rho(\vec{\mathcal{S}}_{G_\rho}(x_0^\rho)) \ \subset \ \mathcal{S}_F(x_0) + \\ e^{\|F\|_\Lambda t}\|x0 - x_0^\rho\| + \rho(\beta + \|F\|_\Lambda\|F\|_\infty)\frac{e^{\|F\|_\Lambda t}-1}{\|F\|_\Lambda} \end{array} \right.$$

Proof — Take any solution $x(\cdot) \in \mathcal{S}_F(x_0)$ to the differential inclusion $x' \in F(x)$, which satisfies $\forall t \geq s \geq 0$,

$$x(t) - x(s) \in \int_s^t F(x(\tau))d\tau$$

Since F is bounded, we deduce that

$$\|x(t) - x(s)\| \leq (t-s)\|F\|_\infty \tag{5.35}$$

On the other hand, since F is Lipschitz,

$$x(t) - x(s) \in (t-s)F(x(s)) + \|F\|_\Lambda \left(\int_s^t \|x(\tau) - x(s)\|d\tau\right) B \tag{5.36}$$

Hence:

$$\forall t \geq s \geq 0,\ x(t) - x(s) \in (t-s)F(x(s)) + \frac{\|F\|_\Lambda\|F\|_\infty}{2}(t-s)^2 B$$

and thus, for $j = 0, \ldots, N-1$,

$$x((j+1)\rho) \in x(j\rho) + \rho F(x(j\rho)) + \alpha\rho^2 B =: \ G_\rho^\alpha((x(j\rho))$$

The sequence $\mathbf{r}_\rho x$ is then a solution to the discrete system G_ρ^α.

Consider now a solution $\vec{x}^\rho$ starting at x_0^ρ to the discrete system G_ρ and its associated piecewise linear interpolation $\mathbf{p}_\rho \vec{x}^\rho$. By the Filippov Theorem, we know that there exists a solution $x_\rho(\cdot) \in \mathcal{S}_F(x_0)$ satisfying inequality

$$\|x_\rho(t) - \mathbf{p}_\rho \vec{x}^\rho(t)\| \le e^{\|F\|_\Lambda t}\left(\|x_0^\rho - x_0\| + \int_0^t d((\mathbf{p}_\rho \vec{x}^\rho)'(s), F(\mathbf{p}_\rho \vec{x}^\rho(s))e^{-\|F\|_\Lambda s} ds\right)$$

But, on each interval $[j\rho, (j+1)\rho[$, we know that

$$(\mathbf{p}_\rho \vec{x}^\rho)'(s) = v_j^\rho := \frac{x_{j+1}^\rho - x_j^\rho}{\rho} \;\&\; \mathbf{p}_\rho \vec{x}^\rho(s) = x_j^\rho + (s - j\rho)v_j^\rho$$

Since $v_j^\rho \in F(x^\rho(j\rho)) + \beta\rho B$ and since F is Lipschitz, we deduce that

$$v_j^\rho \in F(x_j^\rho + (s - j\rho)v_j^\rho) + (\beta + \|F\|_\Lambda \|F\|_\infty)\rho B$$

We thus infer that

$$\|x_\rho(t) - \mathbf{p}_\rho \vec{x}^\rho(t)\| \le e^{\|F\|_\Lambda t}\|x_0^\rho - x_0\| + \rho(\beta + \|F\|_\Lambda \|F\|_\infty)\frac{e^{\|F\|_\Lambda t} - 1}{\|F\|_\Lambda} \;\square$$

5.11 Convergence Theorems

5.11.1 Convergence of Kernels and Basins

Let us denote by $j_T := [\frac{T}{\rho}]$ the integer part of $\frac{T}{\rho}$.

Theorem 5.11.1. *Let us consider a sequence of discretizations G_ρ of a Marchaud map $F : X \rightsquigarrow X$. Then, for every $T \in [0, +\infty]$, the upper limit* $\text{Limsup}_{\rho\to 0+}(L_\rho)$ *of a sequence of subsets L_ρ discretely viable under G_ρ on the interval $[0, [\frac{T}{\rho}]]$ is a closed subset viable under F on the interval $[0, T]$.*

In particular, if K is a closed subset,

$$\begin{cases} \text{Limsup}_{\rho\to 0+}\left(Viab_{G_\rho}^{[\frac{T}{\rho}]}(K)\right) \;\subset\; \text{Viab}_F(K, T) \\ \& \\ \text{Limsup}_{\rho\to 0+}\left(Capt_{G_\rho}^{[\frac{T}{\rho}]}(K)\right) \;\subset\; \text{Capt}_F(K, T) \end{cases}$$

If we assume furthermore that F is Lipschitz and bounded and if

$$G_\rho \;\supset\; \mathbf{1} + \rho F + \frac{\|F\|_\infty \|F\|_\Lambda}{2}\rho^2 B$$

then

$$\begin{cases} \mathrm{Lim}_{\rho\to 0+}\left(Viab_{G_\rho}^{[\frac{T}{\rho}]}(K)\right) = \mathrm{Viab}_F(K,T) \\ \& \\ \mathrm{Lim}_{\rho\to 0+}\left(Capt_{G_\rho}^{[\frac{T}{\rho}]}(K)\right) = \mathrm{Capt}_F(K,T) \end{cases}$$

and in particular,

$$\mathrm{Lim}_{\rho\to 0+}\left(Viab_{G_\rho}(K)\right) = \mathrm{Viab}_F(K)$$

Proof — Let us assume that L_ρ is viable under G_ρ and take any $x_0 = \lim_{\rho\to 0} x_0^\rho$ where $x_0^\rho \in L_\rho$. Then there exists a solution $\vec{x}^\rho := (x_0^\rho, \ldots, x_n^\rho, \ldots)$ to the discrete system G_ρ viable in L_ρ on the interval $[0, [\frac{T}{\rho}]]$, with which we associate the piecewise linear function $x_\rho(\cdot) := \mathbf{p}_\rho \vec{x}^\rho \in \mathcal{C}(0,\infty;X)$ interpolating this sequence at the nodes $n\rho$.

Convergence Theorem 5.10.2 implies that the limit $x(\cdot)$ of some subsequence is a solution to the differential inclusion $x' \in F(x)$. On the other hand, each $t \geq 0$ is the limit of nodes $j_t\rho$, so that $x(t)$ is the limit of $x_\rho(j_t\rho) \in L_\rho$. This implies that $x(t)$ belongs to the upper limit of the subsets L_ρ.

Therefore, the upper limit of the subsets L_ρ viable under G_ρ is a viable under F.

Assume now that F is Lipschitz and bounded and let x_0 belong to the viability kernel $\mathrm{Viab}_F(K)$ of a closed subset K. Then there exists a solution $x(\cdot)$ to the differential inclusion $x' \in F(x)$ viable in K. By Theorem 5.10.4, its image $\mathbf{r}_\rho x$ is a solution to the discrete system G_ρ, which is then viable in K on $[0,T]$. Hence,

$$\mathrm{Viab}_F(K,T) \;\subset\; Viab_{G_\rho}^{[\frac{T}{\rho}]}(K)$$

so that

$$\mathrm{Viab}_F(K,T) \;\subset\; \mathrm{Liminf}_{\rho\to 0+} Viab_{G_\rho}^{[\frac{T}{\rho}]}(K)$$

The limit of the $[\frac{T}{\rho}]$-viability kernels under G_ρ is thus equal to the T-viability kernel under F.

The proof for the capture basins goes along the same lines. □

Proposition 5.11.2. *Let us consider a Lipschitz and bounded map F and a sequence of discretizations of F satisfying*

$$G_\rho \supset \mathbf{1} + \rho F + \frac{\|F\|_\infty \|F\|_\Lambda}{2}\rho^2 B$$

Then the upper limit $\mathrm{Limsup}_{\rho\to 0+}(L_\rho)$ *of a sequence subsets L_ρ invariant under G_ρ on the interval $[0, [\frac{T}{\rho}]]$ is a closed subset invariant under F on $[0,T]$.*

In particular, if K is a closed subset,

$$\mathrm{Limsup}_{\rho\to 0+}\left(Inv_{G_\rho}^{[\frac{T}{\rho}]}(K,T)\right) \subset \mathrm{Inv}_F(K,T)$$

Proof — Let us assume that L_ρ is invariant under G_ρ and take any $x_0 := \lim_{\rho\to 0} x_0^\rho$ where $x_0^\rho \in L_\rho$ in the upper limit $L^\sharp$ of the subsets L_ρ.

Let $x(\cdot)$ any solution to the differential inclusion $x' \in F(x)$. By the Filippov Theorem, there exists a solution $x_\rho(\cdot)$ to this differential inclusion starting at x_0^ρ and satisfying

$$\|x_\rho(t) - x(t)\| \leq e^{\|F\|_\Lambda t}\|x_0 - x_0^\rho\|$$

Theorem 5.10.4 implies that $\mathbf{r}_\rho x_\rho$ is a solution to the discrete system G_ρ. Since K_ρ is invariant under G_ρ, we infer that

$$d(x(\rho j), K) \leq \|x(\rho j) - x_\rho(\rho j)\| \leq \|x_0 - x_0^\rho\| e^{\|F\|_\Lambda \rho j}$$

Each $t \geq 0$ being the limit of nodes $j_t\rho$, we infer that $d(x(t), K) = 0$, i.e., that $x(\cdot)$ is viable in $L^\sharp$. Hence, $L^\sharp$ is invariant under F on $[0,T]$. □

5.11.2 Convergence of Guaranteed Viability Kernels

Theorem 5.11.3. *Let (P,V,c) a Marchaud, Lipschitz and bounded dynamical game.*

Let us consider closed subsets L_ρ enjoying the discrete guaranteed viability property under the discrete dynamical game $(P,V,c)_\rho$.

We assume that there exist λ-Lipschitz selections $\widetilde{p}_\rho$ of the regulation maps

$$\Gamma_\rho(x) := \{p \in P(x) \mid C(x,p,V(x))_\rho \subset L_\rho\}$$

Then the upper limit $L^\sharp$ of the closed subsets L_ρ enjoys the guaranteed viability property under the dynamical game (P, V, c).

Proof — Let us consider λ-Lipschitz selections $\widetilde{p}_\rho$ of the regulation map Γ_ρ which are assumed to exist. By the Ascoli Theorem, we know that the selections $\widetilde{p}_\rho$ remain in a compact subset of $\mathcal{C}(X, Z)$ supplied with the compact convergence topology, so that a subsequence (again denoted by) $\widetilde{p}_\rho$ converges to a Lipschitz selection $\widetilde{p}$ of P uniformly on compact subsets. Let us consider a sequence of elements $x_0^\rho \in L_\rho$ converging to some $x_0 \in L^\sharp$ and let $x(\cdot) \in \mathcal{S}_{c(\cdot,\widetilde{p}(\cdot),V(\cdot))}$ be any solution. We shall prove that it is viable in $L^\sharp$, and thus, that $L^\sharp$ enjoys the guaranteed viability property. Indeed, by the Filippov Theorem, we know that there exists a solution $x_\rho(\cdot) \in \mathcal{S}_{c(\cdot,\widetilde{p}_\rho(\cdot),V(\cdot))}(x_0^\rho)$ satisfying inequality

$$\|x_\rho(t) - x(t)\| \leq e^{c_\lambda t}\left(\|x - x_\rho\| + \int_0^t d(x'(s), c(x(s), \widetilde{p}_\rho(x(s)), V(x(s))))e^{-c_\lambda s} ds\right)$$

Let $Q(t) := x_0 + \|F(x(0))\| \frac{e^{\|F\|_\Lambda t} - 1}{\|F\|_\Lambda} B$ denote the compact set containing all solutions starting from x_0 on the interval $[0, t]$. Then

$$x'(s) \in c(x(s), \widetilde{p}(x(s)), V(x(s))) \ \subset \ c(x(s), \widetilde{p}_\rho(x(s)), V(x(s))) + \eta(s)$$

where $\eta(s) := c_\lambda \sup_{x \in Q(s)} \|\widetilde{p}_\rho(x) - \widetilde{p}(x)\|$ converges to 0. Since $x_\rho(\cdot)$ is a solution to the differential inclusion $x' \in c(x, \tilde{p}_\rho(x), V(x))$, then Theorem 5.10.4 implies that the sequence $\mathbf{r}_\rho x_\rho$ is a solution to the discrete system $C_\rho(x, \tilde{p}_\rho(x), V(x))$ starting at x_0^ρ. On the other hand, L_ρ enjoying the guaranteed viability property, $\mathbf{r}_\rho x_\rho$ is viable in L_ρ. Therefore, for any $j \geq 0$, $x_\rho(j\rho)$ belongs to L_ρ. Each $t \geq 0$ being the limit of nodes $j_t\rho$, then $x(t)$ is the limit of $x^\rho(j_t\rho) \in L_\rho$. This implies that $x(t)$ belongs to the upper limit $L^\sharp$ of the subsets L_ρ.