Modelling The Claims Development Result For Solvency Purposes

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Abstract

We assume that the claims liability process satisfies the distribution-free chain-ladder model assumptions. For claims reserving at time I we predict the total ultimate claim with the information available at time I and, similarly, at time $I+1$ we predict the same total ultimate claim with the (updated) information available at time $I+1$. The claims development result at time $I+1$ for accounting year $(I, I+1]$ is then defined to be the difference between these two successive predictions for the total ultimate claim. In [6, 10] we have analyzed this claims development result and we have quantified its prediction uncertainty. Here, we simplify, modify and illustrate the results obtained in [6, 10]. We emphasize that these results have direct consequences for solvency considerations and were (under the new risk-adjusted solvency regulation) already implemented in industry.

Keywords. Stochastic Claims Reserving, Chain-Ladder Method, Claims Development Result, Loss Experience, Incurred Losses Prior Accident Years, Solvency, Mean Square Error of Prediction.

1. INTRODUCTION

We consider the problem of quantifying the uncertainty associated with the development of claims reserves for prior accident years in general insurance. We assume that we are at time I and we predict the total ultimate claim at time I (with the available information up to time I), and one period later at time $I+1$ we predict the same total ultimate claim with the updated information available at time $I+1$. The difference between these two successive predictions is the so-called claims development result for accounting year $(I, I+1]$. The realization of this claims development result has a direct impact on the profit & loss (P&L) statement and on the financial strength of the insurance company. Therefore, it also needs to be studied for solvency purposes. Here, we analyze the prediction of the claims development result and the possible fluctuations around this prediction (prediction uncertainty). Basically we answer two questions that are of practical relevance:

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(a) In general, one predicts the claims development result for accounting year $(I, I+1]$ in the budget statement at time I by 0. We analyze the uncertainty in this prediction. This is a *prospective view*: "how far can the realization of the claims development result deviate from 0?"
Remark: we discuss below, why the claims development result is predicted by 0.

(b) In the P&L statement at time $I+1$ one then obtains an observation for the claims development result. We analyze whether this observation is within a reasonable range around 0 or whether it is an outlier. This is a *retrospective view*. Moreover, we discuss the possible categorization of this uncertainty.

So let us start with the description of the budget statement and of the P&L statement, for an example we refer to Table 1. The budget values at Jan. 1, year I, are predicted values for the next accounting year $(I, I+1]$. The P&L statement are then the observed values at the end of this accounting year $(I, I+1]$.

Positions a) and b) correspond to the premium income and its associated claims (generated by the premium liability). Position d) corresponds to expenses such as acquisition expenses, head office expenses, etc. Position e) corresponds to the financial returns generated on the balance sheet/assets. All these positions are typically well-understood. They are predicted at Jan. 1, year I (budget values) and one has their observations at Dec. 31, year I in the P&L statement, which describes the financial closing of the insurance company for accounting year $(I, I+1]$.

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	budget values at Jan. 1, year I	P&L statement at Dec. 31, year I
a) premiums earned	4’000’000	4’020’000
b) claims incurred current accident year	-3’200’000	-3’240’000
c) loss experience prior accident years	0	-40’000
d) underwriting and other expenses	-1’000’000	-990’000
e) investment income	600’000	610’000
income before taxes	400’000	360’000

Table 1: Income statement, in \$ 1’000

However, position c), “loss experience prior accident years”, is often much less understood. It corresponds to the difference between the claims reserves at time $t = I$ and at time $t = I + 1$ adjusted for the claim payments during accounting year $(I, I + 1]$ for claims with accident years prior to accounting year I. In the sequel we will denote this position by the **claims development result** (CDR). We analyze this position within the framework of the distribution-free chain-ladder (CL) method. This is described below.

Short-term vs. long-term view

In the classical claims reserving literature, one usually studies the total uncertainty in the claims development until the total ultimate claim is finally settled. For the distribution-free CL method this has first been done by Mack [7]. The study of the total uncertainty of the full run-off is a *long-term view*. This classical view in claims reserving is very important for solving solvency questions, and almost all stochastic claims reserving methods which have been proposed up to now concentrate on this long term view (see Wüthrich-Merz [9]). However, in the present work we concentrate on a second important view, the *short-term view*. The short-term view is important for a variety of reasons:

- If the short-term behaviour is not adequate, the company may simply not get to the "long-term", because it will be declared insolvent before it gets to the long term.

- A short-term view is relevant for management decisions, as actions need to be taken on a regular basis. Note that most actions in an insurance company are usually done on a yearly basis. These are for example financial closings, pricing of insurance products, premium adjustments, etc.

- Reflected through the annual financial statements and reports, the short-term performance of the company is of interest and importance to regulators, clients, investors, rating agencies, stock-markets, etc. Its consistency will ultimately have an impact on the financial strength and the reputation of the company in the insurance market.

Hence our goal is to study the development of the claims reserves on a yearly basis where we assume that the claims development process satisfies the assumptions of the distribution-free chain-ladder model. Our main results, Results 3.1-3.3 and 3.5 below, give an improved version of the results obtained in [6, 10]. De Felice-Moriconi [4] have implemented similar ideas referring to the random variable representing the "Year-End Obligations" of the insurer instead of the CDR. They obtained similar formulas for the prediction error and verified the numerical results with the help of the bootstrap method. They have noticed that their results for aggregated accident years always lie below the analytical ones obtained in [6]. The reason for this is that there is one redundant term in (4.25) of [6]. This is now corrected, see formula (A.4) below. Let us mention that the ideas presented in [6, 10] were already successfully implemented in practice. Prediction error estimates of Year-End Obligations in the overdispersed Poisson model have been derived by ISVAP [5] in a field study on a large sample of Italian MTPL companies. A field study in line with [6, 10] has been published by AISAM-ACME [1]. Moreover, we would also like to mention that during the writing of this paper we have learned that simultaneously similar ideas have been developed by Böhm-Glaab [2].

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2. METHODOLOGY

2.1 Notation

We denote cumulative payments for accident year $i \in \{0, \dots, I\}$ until development year $j \in \{0, \dots, J\}$ by $C_{i,j}$. This means that the ultimate claim for accident year i is given by $C_{i,J}$. For simplicity, we assume that $I = J$ (note that all our results can be generalized to the case $I > J$). Then the outstanding loss liabilities for accident year $i \in \{0, \dots, I\}$ at time $t = I$ are given by

$$R_i^I = C_{i,J} - C_{i,I-i}, \tag{2.1}$$

and at time $t = I + 1$ they are given by

$$R_i^{I+1} = C_{i,J} - C_{i,I-i+1}. \tag{2.2}$$

Let

$$D_I = \left\{C_{i,j};\, i + j \le I \text{ and } i \le I\right\} \tag{2.3}$$

denote the claims data available at time $t = I$ and

$$D_{I+1} = \left\{C_{i,j};\, i + j \le I + 1 \text{ and } i \le I\right\} = D_I \cup \left\{C_{i,I-i+1};\, i \le I\right\} \tag{2.4}$$

denote the claims data available one period later, at time $t = I + 1$. That is, if we go one step ahead in time from I to $I + 1$, we obtain new observations $\left\{C_{i,I-i+1};\, i \le I\right\}$ on the new diagonal of the claims development triangle (cf. Figure 1). More formally, this means that we get an enlargement of the σ-field generated by the observations D_I to the σ-field generated by the observations D_{I+1}, i.e.

$$\sigma(D_I) \to \sigma(D_{I+1}). \tag{2.5}$$

2.2 Distribution-free chain-ladder method

We study the claims development process and the CDR within the framework of the well-known distribution-free CL method. That is, we assume that the cumulative payments $C_{i,j}$ satisfy the assumptions of the distribution-free CL model. The distribution-free CL model has been introduced by Mack [7] and has been used by many other actuaries. It is probably the most popular claims reserving method because it is simple and it delivers, in general, very accurate results.

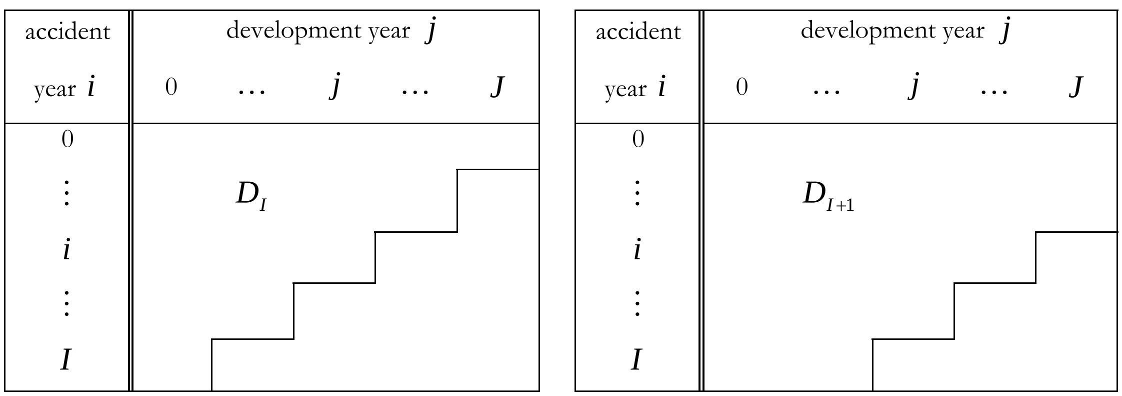

Figure 1: Loss development triangle at time $t = I$ and $t = I + 1$

Model Assumptions 2.1

- Cumulative payments $C_{i,j}$ in different accident years $i \in \{0, \ldots, I\}$ are independent.
- $(C_{i,j})_{j \geq 0}$ are Markov processes and there exist constants $f_j > 0$, $\sigma_j > 0$ such that for all $1 \leq j \leq J$ and $0 \leq i \leq I$ we have

$$E\left[C_{i,j} \,\middle|\, C_{i,j-1}\right] = f_{j-1}\, C_{i,j-1}, \tag{2.6}$$

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$$Var\left(C_{i,j} \mid C_{i,j-1}\right) = \sigma_{j-1}^2 \, C_{i,j-1} \,. \tag{2.7}$$

□

Remarks 2.2

- In the original work of Mack [7] there were weaker assumptions for the definition of the distribution-free CL model, namely the Markov process assumption was replaced by an assumption only on the first two moments (see also Wüthrich-Merz [9]).

- The derivation of an estimate for the estimation error in [10] was done in a time-series framework. This imposes stronger model assumptions. Note also that in (2.7) we require that the cumulative claims $C_{i,j}$ are positive in order to get a meaningful variance assumption.

Model Assumptions 2.1 imply (using the tower property of conditional expectations)

$$E\left[C_{i,J} \mid D_I\right] = C_{i,I-i} \prod_{j=I-i}^{J-1} f_j \qquad \text{and} \qquad E\left[C_{i,J} \mid D_{I+1}\right] = C_{i,I-i+1} \prod_{j=I-i+1}^{J-1} f_j \,. \tag{2.8}$$

This means that for known CL factors f_j we are able to calculate the conditionally expected ultimate claim $C_{i,J}$ given the information D_I and D_{I+1}, respectively.

Of course, in general, the CL factors f_j are not known and need to be estimated. Within the framework of the CL method this is done as follows:

1. At time $t = I$, given information D_I, the CL factors f_j are estimated by

$$\hat{f}_j^I = \frac{\sum_{i=0}^{I-j-1} C_{i,j+1}}{S_j^I}, \qquad \text{where} \qquad S_j^I = \sum_{i=0}^{I-j-1} C_{i,j} \,. \tag{2.9}$$

2. At time $t = I+1$, given information D_{I+1}, the CL factors f_j are estimated by

$$\hat{f}_j^{I+1} = \frac{\sum_{i=0}^{I-j} C_{i,j+1}}{S_j^{I+1}}, \qquad \text{where} \qquad S_j^{I+1} = \sum_{i=0}^{I-j} C_{i,j} \,. \tag{2.10}$$

This means the CL estimates $\hat{f}_j^{I+1}$ at time $I+1$ use the increase in information about the claims development process in the new observed accounting year $(I, I+1]$ and are therefore based on the additional observation $C_{I-j,\,j+1}$.

Mack [7] proved that these are unbiased estimators for f_j and, moreover, that $\hat{f}_j^m$ and $\hat{f}_l^m$ ($m = I$ or $I+1$) are uncorrelated random variables for $j \neq l$ (see Theorem 2 in Mack [7] and Lemma 2.5 in [9]). This implies that, given $C_{i,I-i}$,

$$\hat{C}_{i,j}^I = C_{i,I-i}\ \hat{f}_{I-i}^I \cdots \hat{f}_{j-2}^I\ \hat{f}_{j-1}^I \tag{2.11}$$

is an unbiased estimator for $E\left[C_{i,j} \,\middle|\, D_I\right]$ with $j \geq I-i$ and, given $C_{i,I-i+1}$,

$$\hat{C}_{i,j}^{I+1} = C_{i,I-i+1}\ \hat{f}_{I-i+1}^{I+1} \cdots \hat{f}_{j-2}^{I+1}\ \hat{f}_{j-1}^{I+1} \tag{2.12}$$

is an unbiased estimator for $E\left[C_{i,j} \,\middle|\, D_{I+1}\right]$ with $j \geq I-i+1$.

Remarks 2.3

- The realizations of the estimators $\hat{f}_0^I, \ldots, \hat{f}_{J-1}^I$ are known at time $t = I$, but the realizations of $\hat{f}_0^{I+1}, \ldots, \hat{f}_{J-1}^{I+1}$ are unknown since the observations $C_{I,1}, \ldots, C_{I-J+1,J}$ during the accounting year $(I, I+1]$ are unknown at time I.

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- When indices of accident and development years are such that there are no factor products in (2.11) or (2.12), an empty product is replaced by 1. For example, $\hat{C}^{I}_{i,I-i} = C_{i,I-i}$ and $\hat{C}^{I+1}_{i,I-i+1} = C_{i,I-i+1}$.

- The estimators $\hat{C}^{I+1}_{i,j}$ are based on the CL estimators at time $I+1$ and therefore use the increase in information given by the new observations in the accounting year from I to $I+1$.

2.3 Conditional mean square error of prediction

Assume that we are at time I, that is, we have information D_I and our goal is to predict the random variable $C_{i,J}$. Then, $\hat{C}^{I}_{i,J}$ given in (2.11) is a D_I-measurable predictor for $C_{i,J}$. At time I, we measure the prediction uncertainty with the so-called conditional mean square error of prediction (MSEP) which is defined by

$$msep_{C_{i,J} \mid D_I}\left(\hat{C}^{I}_{i,J}\right) = E\left[\left(C_{i,J} - \hat{C}^{I}_{i,J}\right)^2 \mid D_I\right] \tag{2.13}$$

That is, we measure the prediction uncertainty in the $L^2\left(P\left[\,\cdot \mid D_I\right]\right)$-distance. Because $\hat{C}^{I}_{i,J}$ is D_I-measurable this can easily be decoupled into process variance and estimation error:

$$msep_{C_{i,J} \mid D_I}\left(\hat{C}^{I}_{i,J}\right) = Var\left(C_{i,J} \mid D_I\right) + \left(E\left[C_{i,J} \mid D_I\right] - \hat{C}^{I}_{i,J}\right)^2. \tag{2.14}$$

This means that $\hat{C}^{I}_{i,J}$ is used as predictor for the random variable $C_{i,J}$ and as estimator for the expected value $E\left[C_{i,J} \mid D_I\right]$ at time I. Of course, if the conditional expectation $E\left[C_{i,J} \mid D_I\right]$ is known at time I (i.e. the CL factors f_j are known), it is used as predictor for $C_{i,J}$ and the estimation error term vanishes. For more information on conditional and unconditional MSEP's we refer to Chapter 3 in [9]:

2.4 Claims development result (CDR)

We ignore any prudential margin and assume that claims reserves are set equal to the expected outstanding loss liabilities conditional on the available information at time I and $I+1$, respectively. That is, in our understanding "best estimate" claims reserves correspond to conditional expectations which implies a self-financing property (see Corollary 2.6 in [8]). For known CL factors f_j the conditional expectation $E\left[C_{i,J} \mid D_I\right]$ is known and is therefore used as predictor for $C_{i,J}$ at time I. Similarly, at time $I+1$ the conditional expectation $E\left[C_{i,J} \mid D_{I+1}\right]$ is used as predictor for $C_{i,J}$. Then the **true claims development result** (true CDR) for accounting year $(I, I+1]$ is defined as follows.

Definition 2.4 (True claims development result)

The **true CDR** *for accident year* $i \in \{1, \ldots, I\}$ *in accounting year* $(I, I+1]$ *is given by*

$$
\begin{aligned}
CDR_i(I+1) &= E\left[R_i^I \mid D_I\right] - \left(X_{i,I-i+1} + E\left[R_i^{I+1} \mid D_{I+1}\right]\right) \\
&= E\left[C_{i,J} \mid D_I\right] - E\left[C_{i,J} \mid D_{I+1}\right],
\end{aligned}
\tag{2.15}
$$

where $X_{i,I-i+1} = C_{i,I-i+1} - C_{i,I-i}$ *denotes the incremental payments. Furthermore, the true aggregate is given by*

$$
\sum_{i=1}^{I} CDR_i(I+1). \tag{2.16}
$$

Using the martingale property we see that

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$$E\left[CDR_i(I+1) \mid D_I\right] = 0. \tag{2.17}$$

This means that for known CL factors f_j the expected true CDR (viewed from time I) is equal to zero. Therefore, for known CL factors f_j we refer to $CDR_i(I+1)$ as the **true** CDR. This also justifies the fact that in the budget values of the income statement position c) "loss experience prior accident years" is predicted by $0 (see position c) in Table 1).
The prediction uncertainty of this prediction 0 can then easily be calculated, namely,

$$msep_{CDR_i(I+1)|D_I}(0) = Var\left(CDR_i(I+1) \mid D_I\right) = E\left[C_{i,J} \mid D_I\right]^2 \frac{\sigma_{I-i}^2 / f_{I-i}^2}{C_{i,I-i}}. \tag{2.18}$$

For a proof we refer to formula (5.5) in [10] (apply recursively the model assumptions), and the aggregation of accident years can easily be done because accident years i are independent according to Model Assumptions 2.1.

Unfortunately the CL factors f_j are in general not known and therefore the true CDR is not observable. Replacing the unknown factors by their estimators, i.e., replacing the expected ultimate claims $E\left[C_{i,J} \mid D_I\right]$ and $E\left[C_{i,J} \mid D_{I+1}\right]$ with their estimates $\hat{C}_{i,J}^{I}$ and $\hat{C}_{i,J}^{I+1}$, respectively, the true CDR for accident year i $(1 \le i \le I)$ in accounting year $(I, I+1]$ is predicted/estimated in the CL method by:

Definition 2.5 (Observable claims development result)

The **observable CDR** *for accident year* $i \in \{1, \ldots, I\}$ *in accounting year* $(I, I+1]$ *is given by*

$$\widehat{CDR}_i(I+1) = \hat{R}_i^{D_I} - \left(X_{i,I-i+1} + \hat{R}_i^{D_{I+1}}\right) = \hat{C}_{i,J}^{I} - \hat{C}_{i,J}^{I+1}, \tag{2.19}$$

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where $\hat{R}_i^{D_I}$ *and* $\hat{R}_i^{D_{I+1}}$ *are defined below by* (2.21) *and* (2.22), *respectively. Furthermore, the observable aggregate* CDR *is given by*

$$\sum_{i=1}^{I} C\hat{D}R_i\left(I+1\right). \tag{2.20}$$

Note that under the Model Assumptions 2.1, given $C_{i,I-i}$,

$$\hat{R}_i^{D_I} = \hat{C}_{i,J}^{I} - C_{i,I-i} \qquad \left(1 \le i \le I\right), \tag{2.21}$$

is an unbiased estimator for $E\left[R_i^I \mid D_I\right]$ and, given $C_{i,I-i+1}$,

$$\hat{R}_i^{D_{I+1}} = \hat{C}_{i,J}^{I+1} - C_{i,I-i+1} \qquad \left(1 \le i \le I\right), \tag{2.22}$$

is an unbiased estimator for $E\left[R_i^{I+1} \mid D_{I+1}\right]$.

Remarks 2.6

- We point out the (non-observable) true claims development result $CDR_i\left(I+1\right)$ is approximated by an observable claims development result $C\hat{D}R_i\left(I+1\right)$. In the next section we quantify the quality of this approximation (retrospective view).

- Moreover, the observable claims development result $C\hat{D}R_i\left(I+1\right)$ is the position that occurs in the P&L statement at Dec. 31, year I. This position is in the budget statement predicted by 0. In the next section we also measure the quality of this prediction, which determines the solvency requirements (prospective view).

- We emphasize that such a solvency consideration is only a one-year view. The remaining run-off can, for example, be treated with a cost-of-capital loading that is based on the one-year observable claims development result (this has, for example, been done in the Swiss Solvency Test).

3. MSEP OF THE CLAIMS DEVELOPMENT RESULT

Our goal is to quantify the following two quantities:

$$msep_{\widehat{CDR}_i(I+1)|D_I}(0) = E\left[\left(\widehat{CDR}_i(I+1) - 0\right)^2 \mid D_I\right], \tag{3.1}$$

$$msep_{CDR_i(I+1)|D_I}\left(\widehat{CDR}_i(I+1)\right) = E\left[\left(CDR_i(I+1) - \widehat{CDR}_i(I+1)\right)^2 \mid D_I\right]. \tag{3.2}$$

- The first conditional MSEP gives the prospective solvency point of view. It quantifies the prediction uncertainty in the budget value 0 for the observable claims development result at the end of the accounting period. In the solvency margin we need to hold risk capital for possible negative deviations of $CDR_i(I+1)$ from 0.

- The second conditional MSEP gives a retrospective point of view. It analyzes the distance between the true CDR and the observable CDR. It may, for example, answer the question whether the true CDR could also be positive (if we would know the true CL factors) when we have an observable CDR given by $ -40'000 (see Table 1). Hence, the retrospective view separates pure randomness (process variance) from parameter estimation uncertainties.

In order to quantify the conditional MSEP's we need an estimator for the variance parameters σ_j^2. An unbiased estimate for σ_j^2 is given by (see Lemma 3.5 in [9])

$$\hat{\sigma}_j^2 = \frac{1}{I-j-1} \sum_{i=0}^{I-j-1} C_{i,j} \left(\frac{C_{i,j+1}}{C_{i,j}} - \hat{f}_j \right)^2 . \tag{3.3}$$

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3.1 Single accident year

In this section we give estimators for the two conditional MSEP's defined in (3.1)-(3.2). For their derivation we refer to the appendix. We define

$$\hat{\Delta}_{i,J}^{I} = \frac{\hat{\sigma}_{I-i}^{2} \Big/ \left(\hat{f}_{I-i}^{I}\right)^{2}}{S_{I-i}^{I}} + \sum_{j=I-i+1}^{J-1} \left(\frac{C_{I-j,j}}{S_{j}^{I+1}}\right)^{2} \frac{\hat{\sigma}_{j}^{2} \Big/ \left(\hat{f}_{j}^{I}\right)^{2}}{S_{j}^{I}}, \tag{3.4}$$

$$\hat{\Phi}_{i,J}^{I} = \sum_{j=I-i+1}^{J-1} \left(\frac{C_{I-j,j}}{S_{j}^{I+1}}\right)^{2} \frac{\hat{\sigma}_{j}^{2} \Big/ \left(\hat{f}_{j}^{I}\right)^{2}}{C_{I-j,j}}, \tag{3.5}$$

$$\hat{\Psi}_{i}^{I} = \frac{\hat{\sigma}_{I-i}^{2} \Big/ \left(\hat{f}_{I-i}^{I}\right)^{2}}{C_{i,I-i}} \tag{3.6}$$

and

$$\hat{\Gamma}_{i,J}^{I} = \hat{\Phi}_{i,J}^{I} + \hat{\Psi}_{i}^{I} \geq \hat{\Phi}_{i,J}^{I}. \tag{3.7}$$

We are now ready to give estimators for all the error terms. First of all the variance of the true CDR given in (2.18) is estimated by

$$V\hat{a}r\left(CDR_{i}(I+1)\middle| D_{I}\right) = \left(\hat{C}_{i,J}^{I}\right)^{2} \hat{\Psi}_{i}^{I}. \tag{3.8}$$

The estimator for the conditional MSEP's are then given by:

Result 3.1 (Conditional MSE estimator for a single accident year)

We estimate the conditional MSEP's given in (3.1)-(3.2) *by*

$$\widehat{msep}_{\widehat{CDR}_i(I+1)|D_I}(0) = \left(\widehat{C}_{i,J}^I\right)^2 \left(\widehat{\Gamma}_{i,J}^I + \widehat{\Delta}_{i,J}^I\right), \tag{3.9}$$

$$\widehat{msep}_{\widehat{CDR}_i(I+1)|D_I}\left(\widehat{CDR}_i(I+1)\right) = \left(\widehat{C}_{i,J}^I\right)^2 \left(\widehat{\Phi}_{i,J}^I + \widehat{\Delta}_{i,J}^I\right). \tag{3.10}$$

This immediately implies that we have

$$\begin{aligned}\widehat{msep}_{\widehat{CDR}_i(I+1)|D_I}(0) &= \widehat{msep}_{CDR_i(I+1)|D_I}\left(\widehat{CDR}_i(I+1)\right) + \widehat{Var}\left(CDR_i(I+1)\middle| D_I\right)\\ &\geq \widehat{msep}_{CDR_i(I+1)|D_I}\left(\widehat{CDR}_i(I+1)\right).\end{aligned} \tag{3.11}$$

Note that this is intuitively clear since the true and the observable CDR should move into the same direction according to the observations in accounting year $(I, I+1]$. However, the first line in (3.11) is slightly misleading. Note that we have derived estimators which give an equality on the first line of (3.11). However, this equality holds true only for our estimators where we neglect uncertainties in higher order terms. Note, as already mentioned, for typical real data examples higher order terms are of negligible order which means that we get an approximate equality on the first line of (3.11) (see also derivation in (A.2)). This is similar to the findings presented in Chapter 3 of [9].

3.2 Aggregation over prior accident years

When aggregating over prior accident years, one has to take into account the correlations between different accident years, since the same observations are used to estimate the CL factors and are then applied to different accident years (see also Section 3.2.4 in [9]). Based on the definition of the conditional MSEP for the true aggregate CDR around the aggregated observable CDR the following estimator is obtained:

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Result 3.2 (Conditional MSEP for aggregated accident years, part I)

For aggregated accident years we obtain the following estimator

$$\begin{aligned} & \widehat{msep}_{\sum_{i=1}^{I} CDR_i(I+1)|D_I}\left(\sum_{i=1}^{I} \widehat{CDR}_i(I+1)\right) \\ & \quad = \sum_{i=1}^{I} \widehat{msep}_{CDR_i(I+1)|D_I}\left(\widehat{CDR}_i(I+1)\right) + 2\sum_{k>i>0} \hat{C}_{i,J}^{I}\, \hat{C}_{k,J}^{I}\left(\hat{\Phi}_{i,J}^{I} + \hat{\Lambda}_{i,J}^{I}\right) \end{aligned} \tag{3.12}$$

with

$$\hat{\Lambda}_{i,J}^{I} = \frac{C_{i,I-i}}{S_{I-i}^{I+1}} \frac{\hat{\sigma}_{I-i}^{2} \Big/ \left(\hat{f}_{I-i}^{I}\right)^2}{S_{I-i}^{I}} + \sum_{j=I-i+1}^{J-1} \left(\frac{C_{I-j,j}}{S_j^{I+1}}\right)^2 \frac{\hat{\sigma}_j^2 \Big/ \left(\hat{f}_j^{I}\right)^2}{S_j^{I}}. \tag{3.13}$$

For the conditional MSEP of the aggregated observable CDR around 0 we need an additional definition.

$$\hat{\Xi}_{i,J}^{I} = \hat{\Phi}_{i,J}^{I} + \frac{\hat{\sigma}_{I-i}^{2} \Big/ \left(\hat{f}_{I-i}^{I}\right)^2}{S_{I-i}^{I+1}} \geq \hat{\Phi}_{i,J}^{I}. \tag{3.14}$$

Result 3.3 (Conditional MSEP for aggregated accident years, part II)

For aggregated accident years we obtain the following estimator

$$\begin{aligned} & \widehat{msep}_{\sum_{i=1}^{I} \widehat{CDR}_i(I+1)|D_I}(0) \\ & \quad = \sum_{i=1}^{I} \widehat{msep}_{\widehat{CDR}_i(I+1)|D_I}(0) + 2\sum_{k>i>0} \hat{C}_{i,J}^{I}\, \hat{C}_{k,J}^{I}\left(\hat{\Xi}_{i,J}^{I} + \hat{\Lambda}_{i,J}^{I}\right). \end{aligned} \tag{3.15}$$

Note that (3.15) can be rewritten as follows:

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$$\begin{aligned} &\hat{msep}_{\sum_{i=1}^{I} \hat{CDR}_i(I+1) | D_I}(0) \qquad (3.16) \\ &\quad = \hat{msep}_{\sum_{i=1}^{I} CDR_i(I+1) | D_I}\left(\sum_{i=1}^{I} \hat{CDR}_i(I+1)\right) \\ &\qquad + \sum_{i=1}^{I} \hat{Var}\left(CDR_i(I+1) \middle| D_I\right) + 2 \sum_{k>i>0} \hat{C}_{i,J}^I \hat{C}_{k,J}^I (\hat{\Xi}_{i,J}^I - \hat{\Phi}_{i,J}^I) \\ &\quad \geq \hat{msep}_{\sum_{i=1}^{I} CDR_i(I+1) | D_I}\left(\sum_{i=1}^{I} \hat{CDR}_i(I+1)\right). \end{aligned}$$

Hence, we obtain the same decoupling for aggregated accident years as for single accident years.

Remarks 3.4 (Comparison to the classical Mack [7] formula)

In Results 3.1-3.3 we have obtained a natural split into process variance and estimation error. However, this split has no longer this clear distinction as it appears. The reason is that the process variance also influences the volatility of $\hat{f}_j^{I+1}$ and hence is part of the estimation error. In other approaches one may obtain other splits, e.g. in the credibility chain ladder method (see Bühlmann et al. [3]) one obtains a different split. Therefore we modify Results 3.1.-3.3 which leads to a formula that gives interpretations in terms of the classical Mack [7] formula, see also (4.2)-(4.3) below.

Result 3.5

For single accident years we obtain from Result 3.1

$$\widehat{msep}_{\widehat{CDR}_i(I+1)|D_I}(0) = \left(\widehat{C}_{i,J}^I\right)^2 \left(\widehat{\Gamma}_{i,J}^I + \widehat{\Delta}_{i,J}^I\right) \tag{3.17}$$

$$= \left(\widehat{C}_{i,J}^I\right)^2 \left(\frac{\widehat{\sigma}_{I-i}^2 / \left(\widehat{f}_{I-i}^I\right)^2}{C_{i,I-i}} + \frac{\widehat{\sigma}_{I-i}^2 / \left(\widehat{f}_{I-i}^I\right)^2}{S_{I-i}^I} + \sum_{j=I-i+1}^{J-1} \frac{C_{I-j,j}}{S_j^{I+1}} \frac{\widehat{\sigma}_j^2 / \left(\widehat{f}_j^I\right)^2}{S_j^I} \right).$$

For aggregated accident years we obtain from Result 3.3

$$\widehat{msep}_{\sum_{i=1}^{I} \widehat{CDR}_i(I+1)|D_I}(0) = \sum_{i=1}^{I} \widehat{msep}_{\widehat{CDR}(I+1)|D_I}(0) \tag{3.18}$$

$$+ 2 \sum_{k>i>0} \widehat{C}_{i,J}^I \widehat{C}_{k,J}^I \left[\frac{\widehat{\sigma}_{I-i}^2 / \left(\widehat{f}_{I-i}^I\right)^2}{S_{I-i}^I} + \sum_{j=I-i+1}^{J-1} \frac{C_{I-j,j}}{S_j^{I+1}} \frac{\widehat{\sigma}_j^2 / \left(\widehat{f}_j^I\right)^2}{S_j^I} \right].$$

We compare this now to the classical Mack [7] formula. For single accident years the conditional MSEP of the predictor for the ultimate claim is given in Theorem 3 in Mack [7] (see also Estimator 3.12 in [9]). We see from (3.17) that the conditional MSEP of the CDR considers only the first term of the process variance of the classical Mack [7] formula $(j = I - i)$ and for the estimation error the next diagonal is fully considered $(j = I - i)$ but all remaining runoff cells $(j \geq I - i + 1)$ are scaled by $C_{i,I-i} / S_j^{I+1} \leq 1$. For aggregated accident years the conditional MSEP of the predictor for the ultimate claim is given on page 220 in Mack [7] (see also Estimator 3.16 in [9]). We see from (3.18) that the conditional MSEP of the CDR for aggregated accident years considers the estimation error for the next accounting year $(j = I - i)$ and all other accounting years $(j \geq I - i + 1)$ are scaled by $C_{i,I-i} / S_j^{I+1} \leq 1$.

Hence we have obtained a different split that allows for easy interpretations in terms of the Mack [7] formula. However, note that these interpretations only hold true for linear approximations (A.1), otherwise the picture is more involved.

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4. NUMERICAL EXAMPLE AND CONCLUSIONS

For our numerical example we use the dataset given in Table 2. The table contains cumulative payments $C_{i,j}$ for accident years $i \in \{0,1,\ldots,8\}$ at time $I=8$ and at time $I+1=9$. Hence this allows for an explicitly calculation of the observable claims development result.

	$j=0$	1	2	3	4	5	6	7	8
$i=0$	2'202'584	3'210'449	3'468'122	3'545'070	3'621'627	3'644'636	3'669'012	3'674'511	3'678'633
$i=1$	2'350'650	3'553'023	3'783'846	3'840'067	3'865'187	3'878'744	3'898'281	3'902'425	3'906'738
$i=2$	2'321'885	3'424'190	3'700'876	3'798'198	3'854'755	3'878'993	3'898'825	3'902'130	
$i=3$	2'171'487	3'165'274	3'395'841	3'466'453	3'515'703	3'548'422	3'564'470		
$i=4$	2'140'328	3'157'079	3'399'262	3'500'520	3'585'812	3'624'784			
$i=5$	2'290'664	3'338'197	3'550'332	3'641'036	3'679'909				
$i=6$	2'148'216	3'219'775	3'428'335	3'511'860					
$i=7$	2'143'728	3'158'581	3'376'375						
$i=8$	2'144'738	3'218'196							
$\hat{f}_j^I$	1.4759	1.0719	1.0232	1.0161	1.0063	1.0056	1.0013	1.0011	
$\hat{f}_j^{I+1}$	1.4786	1.0715	1.0233	1.0152	1.0072	1.0053	1.0011	1.0011	
$\hat{\sigma}_j^2$	911.43	189.82	97.81	178.75	20.64	3.23	0.36	0.04	

Table 2: Run-off triangle (cumulative payments, in $ 1'000) for time $I=8$ and $I=9$

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Table 2 summarizes the CL estimates $\hat{f}_j^I$ and $\hat{f}_j^{I+1}$ of the age-to-age factors f_j as well as the variance estimates $\hat{\sigma}_j^2$ for $j = 0,\ldots,7$. Since we do not have enough data to estimate σ_7^2 (recall $I = J$) we use the extrapolation given in Mack [7]:

$$\hat{\sigma}_7^2 = \min\left\{\hat{\sigma}_6^2, \hat{\sigma}_5^2, \hat{\sigma}_6^4 / \hat{\sigma}_5^2\right\}. \tag{4.1}$$

Using the estimates $\hat{f}_j^I$ and $\hat{f}_j^{I+1}$ we calculate the claims reserves $\hat{R}_i^{D_I}$ for the outstanding claims liabilities R_i^I at time $t = I$ and $X_{i,I-i+1} + \hat{R}_i^{D_{I+1}}$ for $X_{i,I-i+1} + R_i^{I+1}$ at time $t = I+1$, respectively. This then gives realizations of the observable CDR for single accident years and for aggregated accident years (see Table 3). Observe that we have a negative observable aggregate CDR at time $I+1$ of about \$ -40'000 (which corresponds to position c) in the P&L statement in Table 1).

i	$\hat{R}_i^{D_I}$	$X_{i,I-i+1} + \hat{R}_i^{D_{I+1}}$	$\widehat{CDR}_i(I+1)$
0	0	0	0
1	4'378	4'313	65
2	9'348	7'649	1'698
3	28'392	24'046	4'347
4	51'444	66'494	-15'050
5	111'811	93'451	18'360
6	187'084	189'851	-2'767
7	411'864	401'134	10'731
8	1'433'505	1'490'962	-57'458
Total	2'237'826	2'277'900	-40'075

Table 3: Realization of the observable CDR at time $t = I+1$, in \$ 1'000

The question which we now have is whether the true aggregate CDR could also be positive if we had known the true CL factors f_j at time $t = I$ (retrospective view). We therefore perform the variance and MSEP analysis using the results of Section 3. Table 4 provides the estimates for single and aggregated accident years.

On the other hand we would like to know, how this observation of $ -40'000 corresponds to the prediction uncertainty in the budget values, where we have predicted that the CDR is $ 0 (see position c) in Table 1). This is the prospective (solvency) view.

We observe that the estimated standard deviation of the true aggregate CDR is equal to $ 65'412, which means that it is not unlikely to have the true aggregate CDR in the range of about $ $\pm$40'000. Moreover, we see that the square root of the estimate for the MSEP between true and observable CDR is of size $ 33'856 (see Table 4), this means that it is likely that the true CDR has the same sign as the observable CDR which is $ -40'000. Therefore also the knowledge of the true CL factors would probably have led to a negative claims development experience.

Moreover, note that the prediction 0 in the budget values has a prediction uncertainty relative to the observable CDR of $ 81'080 which means that it is not unlikely to have an observable CDR of $ -40'000. In other words the solvency capital/risk margin for the CDR should directly be related to this value of $ 81'080.

i	$\hat{R}_i^{D_I}$	$\hat{Var}^{1/2}$	$\hat{msep}_{CDR\mid D_I}\left(\hat{CDR}\right)^{1/2}$	$\hat{msep}_{\hat{CDR}\mid D_I}(0)^{1/2}$	$msep_{Mack}^{1/2}$
0	0				
1	4'378	395	407	567	567
2	9'348	1'185	900	1'488	1'566
3	28'392	3'395	1'966	3'923	4'157
4	51'444	8'673	4'395	9'723	10'536
5	111'811	25'877	11'804	28'443	30'319
6	187'084	18'875	9'100	20'954	35'967
7	411'864	25'822	11'131	28'119	45'090
8	1'433'505	49'978	18'581	53'320	69'552
$\mathrm{cov}^{1/2}$		0	20'754	39'746	50'361
Total	2'237'826	65'412	33'856	81'080	108'401

Table 4: Volatilities of the estimates in $ 1'000 with:

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$\hat{R}_i^{D_I}$	estimated reserves at time $t = I$, cf. (2.21)
$V\hat{a}r^{1/2}$	estimated std. dev. of the true CDR, cf. (3.8)
$m\hat{s}ep_{\hat{CDR}\mid D_I}\left(\hat{CDR}\right)^{1/2}$	estimated $msep^{1/2}$ between true and observable CDR, cf. (3.10) and (3.12)
$m\hat{s}ep_{\hat{CDR}\mid D_I}(0)^{1/2}$	prediction std. dev. of 0 compared to $\hat{CDR}_i(I+1)$, cf. (3.9) and (3.15)
$msep_{Mack}^{1/2}$	$msep^{1/2}$ of the ultimate claim, cf. Mack [7] and (4.3)

Note that we only consider the one-year uncertainty of the claims reserves run-off. This is exactly the short term view/picture that should look fine to get to the long term. In order to treat the full run-off one can then add, for example, a cost-of-capital margin to the remaining run-off which ensures that the future solvency capital can be financed. We emphasize that it is important to add a margin which ensures the smooth run-off of the whole liabilities after the next accounting year.

Finally, these results are compared to the classical Mack formula [7] for the estimate of the conditional MSEP of the ultimate claim $C_{i,J}$ by $\hat{C}_{i,J}^I$ in the distribution-free CL model. The Mack formula [7] gives the total uncertainty of the full run-off (long term view) which estimates

$$msep_{Mack}\left(\hat{C}_{i,J}^I\right) = E\left[\left(C_{i,J} - \hat{C}_{i,J}^I\right)^2 \mid D_I\right] \tag{4.2}$$

and

$$msep_{Mack}\left(\sum_{i=1}^I \hat{C}_{i,J}^I\right) = E\left[\left(\sum_{i=1}^I C_{i,J} - \sum_{i=1}^I \hat{C}_{i,J}^I\right)^2 \mid D_I\right], \tag{4.3}$$

see also Estimator 3.16 in [9]. Notice that the information in the next accounting year (diagonal $I+1$) contributes substantially to the total uncertainty of the total ultimate claim over prior accident years. That is, the uncertainty in the next accounting year is \$ 81'080 and

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the total uncertainty is \$ 108'401. Note that we have chosen a short-tailed line of business so it is clear that a lot of uncertainty is already contained in the next accounting year. Generally speaking, the portion of uncertainty which is already contained in the next accounting year is larger for short-tailed business than for long-tailed business since in long-tailed business the adverse movements in the claims reserves emerge slowly over many years. If one chooses long-tailed lines of business then the one-year risk is about 2/3 of the full run-off risk. This observation is inline with a European field study in different companies, see AISAM-ACME [1].

APPENDIX A. PROOFS AND DERIVATIONS

Assume that a_j are positive constants with $1 >> a_j$ then we have

$$\prod_{j=1}^{J}\left(1+a_j\right)-1 \approx \sum_{j=1}^{J} a_j \ , \tag{A.1}$$

where the right-hand side is a lower bound for the left-hand side. Using the above formula we will approximate all product terms from our previous work [10] by summations.

Derivation of Result 3.1. We first give the derivation of Result 3.1 for a single accident year. Note that the term $\hat{\Delta}_{i,J}^{I}$ is given in formula (3.10) of [10]. Henceforth there remains to derive the terms $\hat{\Phi}_{i,J}^{I}$ and $\hat{\Gamma}_{i,J}^{I}$.

For the term $\hat{\Phi}_{i,J}^{I}$ we obtain from formula (3.9) in [10]

$$\left[1+\frac{\hat{\sigma}_{I-i}^{2}\Big/\left(\hat{f}_{I-i}^{I}\right)^{2}}{C_{i,I-i}}\right]\left(\prod_{j=I-i+1}^{J-1}\left(1+\frac{\hat{\sigma}_{j}^{2}\Big/\left(\hat{f}_{j}^{I}\right)^{2}}{C_{I-j,j}}\left(\frac{C_{I-j,j}}{S_{j}^{I+1}}\right)^{2}\right)-1\right)$$

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$$\approx \left[1+\frac{\hat{\sigma}_{I-i}^2 \big/ \left(\hat{f}_{I-i}^I\right)^2}{C_{i,I-i}}\right] \sum_{j=I-i+1}^{J-1} \frac{\hat{\sigma}_j^2 \big/ \left(\hat{f}_j^I\right)^2}{C_{I-j,j}} \left(\frac{C_{I-j,j}}{S_j^{I+1}}\right)^2 \tag{A.2}$$

$$\approx \sum_{j=I-i+1}^{J-1} \frac{\hat{\sigma}_j^2 \big/ \left(\hat{f}_j^I\right)^2}{C_{I-j,j}} \left(\frac{C_{I-j,j}}{S_j^{I+1}}\right)^2 = \hat{\Phi}_{i,J}^I ,$$

where the approximations are accurate because $1 >> \frac{\hat{\sigma}_{I-i}^2 \big/ \left(\hat{f}_{I-i}^I\right)^2}{C_{i,I-i}}$ for typical claims reserving data.

For the term $\hat{\Gamma}_{i,J}^I$ we obtain from (3.16) in [10]

$$\left(\left[1+\frac{\hat{\sigma}_{I-i}^2 \big/ \left(\hat{f}_{I-i}^I\right)^2}{C_{i,I-i}}\right] \prod_{j=I-i+1}^{J-1} \left(1+\frac{\hat{\sigma}_j^2 \big/ \left(\hat{f}_j^I\right)^2}{C_{I-j,j}} \left(\frac{C_{I-j,j}}{S_j^{I+1}}\right)^2\right)\right) - 1 \tag{A.3}$$

$$\approx \frac{\hat{\sigma}_{I-i}^2 \big/ \left(\hat{f}_{I-i}^I\right)^2}{C_{i,I-i}} + \sum_{j=I-i+1}^{J-1} \frac{\hat{\sigma}_j^2 \big/ \left(\hat{f}_j^I\right)^2}{C_{I-j,j}} \left(\frac{C_{I-j,j}}{S_j^{I+1}}\right)^2 = \hat{\Psi}_i^I + \hat{\Phi}_{i,J}^I = \hat{\Gamma}_{i,J}^I .$$

Henceforth, Result 3.1 is obtained from (3.8), (3.14) and (3.15) in [10].

□

Derivations of Results 3.2 and 3.3. We now turn to Result 3.2. All that remains to derive are the correlation terms.

We start with the derivation of $\hat{\Lambda}_{k,J}^I$ (this differs from the calculation in [6]). From (4.24)-(4.25) in [6] we see that for $i < k$ the cross covariance term of the estimation error

$$C_{i,I-i}^{-1}\, C_{k,I-k}^{-1}\, E\left[\hat{CDR}_i(I+1) \middle| D_I\right] E\left[\hat{CDR}_k(I+1) \middle| D_I\right]$$

is estimated by resampled values $\hat{f}_j$, given D_I, which implies

$$E\left[\left(\prod_{j=I-i}^{J-1}\hat{f}_j^I - f_{I-i}\prod_{j=I-i+1}^{J-1}\left(\frac{S_j^I}{S_j^{I+1}}\hat{f}_j^I + f_j\frac{C_{I-j,j}}{S_j^{I+1}}\right)\right)\right.$$

$$\left.\times\left(\prod_{j=I-k}^{J-1}\hat{f}_j^I - f_{I-k}\prod_{j=I-k+1}^{J-1}\left(\frac{S_j^I}{S_j^{I+1}}\hat{f}_j^I + f_j\frac{C_{I-j,j}}{S_j^{I+1}}\right)\right)\middle| D_I\right] \tag{A.4}$$

$$=\left(\prod_{j=I-i}^{J-1}E\left[\left(\hat{f}_j^I\right)^2 \mid D_I\right] + f_{I-i}^2\prod_{j=I-i+1}^{J-1}E\left[\left(\frac{S_j^I}{S_j^{I+1}}\hat{f}_j^I + f_j\frac{C_{I-j,j}}{S_j^{I+1}}\right)^2 \mid D_I\right]\right.$$

$$- f_{I-i}^2\prod_{j=I-i+1}^{J-1}E\left[\hat{f}_j^I\left(\frac{S_j^I}{S_j^{I+1}}\hat{f}_j^I + f_j\frac{C_{I-j,j}}{S_j^{I+1}}\right)\middle| D_I\right]$$

$$\left.-\prod_{j=I-i}^{J-1}E\left[\hat{f}_j^I\left(\frac{S_j^I}{S_j^{I+1}}\hat{f}_j^I + f_j\frac{C_{I-j,j}}{S_j^{I+1}}\right)\middle| D_I\right]\right)\prod_{j=I-k}^{I-i-1}f_j\ .$$

Note that the last two lines differ from (4.25) in [6]. This last expression is now equal to (see also Section 4.1.2 in [6])

$$=\left\{\prod_{j=I-i}^{J-1}\left(\frac{\sigma_j^2}{S_j^I} + f_j^2\right) + f_{I-i}^2\prod_{j=I-i+1}^{J-1}\left(\left(\frac{S_j^I}{S_j^{I+1}}\right)^2\frac{\sigma_j^2}{S_j^I} + f_j^2\right)\right.$$

$$\left.- f_{I-i}^2\prod_{j=I-i+1}^{J-1}\left(\frac{S_j^I}{S_j^{I+1}}\frac{\sigma_j^2}{S_j^I} + f_j^2\right) - \prod_{j=I-i}^{J-1}\left(\frac{S_j^I}{S_j^{I+1}}\frac{\sigma_j^2}{S_j^I} + f_j^2\right)\right\}\prod_{j=I-k}^{I-i-1}f_j\ .$$

Next we use (A.1), so we see that the last line can be approximated by

$$\approx\left\{\sum_{j=I-i}^{J-1}\frac{\sigma_j^2\big/f_j^2}{S_j^I} + \sum_{j=I-i+1}^{J-1}\left(\frac{S_j^I}{S_j^{I+1}}\right)^2\frac{\sigma_j^2\big/f_j^2}{S_j^I}\right.$$

$$\left.-\sum_{j=I-i+1}^{J-1}\frac{S_j^I}{S_j^{I+1}}\frac{\sigma_j^2\big/f_j^2}{S_j^I} - \sum_{j=I-i}^{J-1}\frac{S_j^I}{S_j^{I+1}}\frac{\sigma_j^2\big/f_j^2}{S_j^I}\right\}\prod_{j=I-k}^{I-i-1}f_j\prod_{j=I-i}^{J-1}f_j^2$$

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$$=\left\{\frac{\sigma_{I-i}^2 \big/ f_{I-i}^2}{S_{I-i}^I}-\frac{S_{I-i}^I}{S_{I-i}^{I+1}}\frac{\sigma_{I-i}^2 \big/ f_{I-i}^2}{S_{I-i}^I}+\sum_{j=I-i+1}^{J-1}\left(1-\frac{S_j^I}{S_j^{I+1}}\right)^2\frac{\sigma_j^2 \big/ f_j^2}{S_j^I}\right\}\prod_{j=I-k}^{I-i-1} f_j \prod_{j=I-i}^{J-1} f_j^2 .$$

Next we note that $1-S_j^I \big/ S_j^{I+1}=C_{I-j,j} \big/ S_j^{I+1}$ hence this last term is equal to

$$=\left\{\frac{C_{i,I-i}}{S_{I-i}^{I+1}}\frac{\sigma_{I-i}^2 \big/ f_{I-i}^2}{S_{I-i}^I}+\sum_{j=I-i+1}^{J-1}\left(\frac{C_{I-j,j}}{S_j^{I+1}}\right)^2\frac{\sigma_j^2 \big/ f_j^2}{S_j^I}\right\}\prod_{j=I-k}^{I-i-1} f_j \prod_{j=I-i}^{J-1} f_j^2 .$$

Hence plugging in the estimators for f_j and σ_j^2 at time I yields the claim.

Hence there remains to calculate the second term in Result 3.2. From (3.13) in [10] we again obtain the claim, using that $1 >> \frac{\hat{\sigma}_{I-i}^2 \big/ \left(\hat{f}_{I-i}^I\right)^2}{C_{i,I-i}}$ for typical claims reserving data.

So there remains to derive Result 3.3. The proof is completely analogous, the term containing $\hat{\Lambda}_{i,J}^I$ was obtained above. The term $\hat{\Xi}_{i,J}^I$ is obtained from (3.17) in [10] analogous to (A.3).

This completes the derivations.

$\square$

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