OPTIMIZATION OF THE NON-LIFE INSURANCE RISK DIVERSIFICATION IN SOLVENCY II

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Abstract

According to the current Solvency II standard approach, non-life risk capital charges take into account geographical diversification by adjusting volume measures using a Herfindahl-Hirschman concentration index for premiums and reserves at a line of business level. The lower the Herfindahl index the less concentrated is a portfolio and the greater is its diversification extent. The diversification factor of a portfolio of risks with respect to some risk measure is defined to be the quotient of the portfolio risk measure to the sum of the stand-alone risk measures over all risks in the portfolio. Maximum diversification is obtained by minimizing the diversification factor. According to the QIS4 proposal the minimum diversification factor is equal to 0.75. This value is not optimal. If the risk measure is proportional to the standard deviation of the risk, then the absolute minimum value of 0.707 allows for an additional diversification reduction of maximum magnitude 4.3%. The latter is true in the case of the value-at-risk and the conditional value-at-risk measures for the class of multivariate elliptical risk distributions. However, the current Solvency II standard approach to non-life risk relies on log-normal distributions. In this framework, the minimum diversification factor, which depends on the volatility of the portfolio, is in the average equal to 0.667, which results in an absolute diversification reduction of magnitude 8.3% compared to QIS4. Extending the analysis to the class of multivariate log-elliptical risk distributions, further results on the minimum diversification factor can be obtained. For the class of multivariate log-Laplace distributions, which are able to model fat tails similarly to the class of generalized Pareto distributions in Extreme Value Theory, this minimum value is in the average 0.68 resulting in an absolute reduction of lower magnitude 7%.

Key words

Solvency II non-life risk, value-at-risk, conditional value-at-risk, Herfindahl-Hirschman index, diversification factor, multivariate elliptical and log-elliptical risk distributions

1. Introduction

Though an old idea, the measurement and allocation of diversification in portfolios of asset and/or liability risks is a difficult problem, which has found so far many answers. The *diversification effect* of a portfolio of risks is the difference between the sum of the risk measures of stand-alone risks in the portfolio and the risk measure of all risks in the portfolio taken together, which is typically non-negative, at least for positive dependent risks. The *risk allocation problem* consists to apportion the diversification effect to the risks of a portfolio in a fair manner, to obtain new risk measures of the risks of a portfolio. The first mathematical approach to diversification is due to Markowitz(1952/59/87/94), whose classical portfolio selection model applies to the efficient diversification of investments. The present paper considers only the diversification effect of a portfolio of non-life risks. According to the current Solvency II standard approach, which is specified in QIS4(2008), non-life risk capital charges take into account geographical diversification by adjusting volume measures using a Herfindahl-Hirschman concentration index for premiums and reserves at a line of business level. The lower the Herfindahl index the less concentrated is a portfolio and the greater is its diversification extent. While from a theoretical point of view the link between diversification and concentration has been somewhat studied in Foldvary(2006), the present contribution focuses on the practical relevance of diversification in the Solvency II project.

The *diversification factor* of a portfolio of risks with respect to some risk measure is defined to be the quotient of the portfolio risk measure to the sum of the stand-alone risk measures over all risks in the portfolio. Maximum diversification is obtained by minimizing the diversification factor. Observe that the greater the diversification reduction is, the less risk capital is needed and the more new business can be written. Therefore optimal diversification has an important practical relevance. According to the QIS4 proposal the minimum diversification factor is equal to 0.75. This value is not optimal. If the risk measure is proportional to the standard deviation of the risk, then the absolute minimum value of 0.707 allows for an additional diversification reduction of maximum magnitude 4.3%. The latter is true in the case of the value-at-risk and the conditional value-at-risk measures for the class of multivariate elliptical risk distributions. However, the current Solvency II standard approach to non-life risk relies on log-normal distributions. Under this assumption, the minimum diversification factor, which depends on the volatility of the portfolio, is in the average equal to 0.667, which results in an absolute diversification reduction of magnitude 8.3% compared to QIS4. Extending the analysis to the class of multivariate log-elliptical risk distributions, further results on the minimum diversification factor can be obtained. For the class of multivariate log-Laplace distributions, which are able to model fat tails similarly to the class of generalized Pareto distributions in Extreme Value Theory, this minimum value is in the average 0.68 resulting in an absolute reduction of lower magnitude 7%.

A more detailed account of the content follows. Section 2 reviews the Solvency II standard approach to non-life risks and presents a simple explanation for the proposed diversification factor, which is missing in QIS4(2008). It is based on the *intra-portfolio correlation* coefficient. Section 3 derives the minimum value of the diversification factor for risk measures proportional to the standard deviation of the risks. Typically, the obtained result applies to the class of multivariate elliptical distributions. A rigorous approach to the current standard Solvency II approach is found in Section 4, where minimum diversification factors are derived for the class of multivariate log-normal distributions. Section 5 extends the results of Section 4 to multivariate log-elliptical distributions, and exemplifies the results for the class of multivariate log-Laplace distributions. Finally, Section 6 illustrates the numerical impact of our findings on the current Solvency II standard approach.

2. Solvency II non-life risk diversification according to QIS4

Recall the simple actuarial rationale for the non-life economic capital formula proposed for Solvency II in QIS3(2007), which has been presented in Hürlimann(2008a).

Suppose an insurance risk portfolio over a fixed time period, say over a one-year time period $[0,1]$ between the times $t=0$ and $t=1$, is described by the following quantities:

P : the (net) *risk premium* of the portfolio for the time period
S : the random *aggregate claims* of the portfolio over the time period

While the risk premium is supposed to be known at the beginning of the period, the random aggregate claims are not. The *random loss* of the portfolio at the beginning of the time period is described by the difference between aggregate claims and risk premium and defined by

$$L = S - P. \tag{2.1}$$

In non-life insurance the aggregate claims over the time period are taken exclusive of the "run-off" and include the claims Y paid out during the time period and the change in claims reserves $\Delta R = R_1 - R_0$, where R_t denotes the *claims reserves* at time t, which consists of the total reserves for outstanding claims and for IBNR claims. Therefore one has the equality $S = Y + \Delta R$. At time $t=0$ the claims reserve R_0 is known while R_1 is unknown. The *volume* $V = P + R_0$ of the portfolio, which is defined as the sum of the risk premium and the claims reserves at the beginning of the period, is known at time $t=0$. Consider the ratio of the random loss to the volume, which can be written as

$$\frac{L}{V} = \frac{Y + R_1 - (P + R_0)}{P + R_0} = X - 1, \quad X = \frac{Y + R_1}{P + R_0} \tag{2.2}$$

where X represents a *combined ratio* of the portfolio (ratio of incurred claims inclusive "run-off" to the premium and reserve volume). The actuarial equivalence principle or fair value principle $E[L]=0$ implies that $E[X]=1$. The Solvency II model assumes that X is log-normally distributed, say with parameters μ_X and σ_X. With $\sigma = \sqrt{Var[X]}$ one has

$$\mu_X = -\tfrac{1}{2}\sigma_X^2, \quad \sigma_X^2 = \ln(1+\sigma^2). \tag{2.3}$$

The *economic capital* of the insurance risk portfolio to the confidence level α is supposed to depend only on the random loss and is denoted by $EC_\alpha[L]$. In the standard Solvency II approach, the economic capital is defined to be the value-at-risk (VaR) of the random loss taken at the confidence level $\alpha = 99.5\%$. Using (2.2), the log-normal assumption on X and (2.3) one derives the non-life economic capital formula

$$EC_\alpha[L] = VaR_\alpha[L] = \rho_\alpha(\sigma)\cdot V \tag{2.4}$$

with

$$\rho_\alpha(\sigma) = \frac{\exp\left\{\Phi^{-1}(\alpha)\cdot\sqrt{\ln(1+\sigma^2)}\right\}}{\sqrt{1+\sigma^2}} - 1, \tag{2.5}$$

where $\Phi^{-1}(\alpha)$ denotes the α-quantile of the standard normal distribution $\Phi(x)$. Alternatively, and as first suggested in the CEIOPS consultation paper CP20(2006), 5.309, p.137, one can instead define the economic capital to be the tail value-at-risk (TailVaR) or conditional value-at-risk (CVaR) of the random loss taken at the confidence level $\alpha = 99\%$. With this choice of risk measure, one obtains the following economic capital formula

$$EC_\alpha[L] = CVaR_\alpha[L] = \rho_\alpha(\sigma) \cdot V \tag{2.6}$$

with

$$\rho_\alpha(\sigma) = \frac{\alpha - \Phi\left(\Phi^{-1}(\alpha) - \sqrt{\ln\left(1+\sigma^2\right)}\right)}{1-\alpha}. \tag{2.7}$$

As a novel feature QIS4(2008) takes into account geographical diversification by adjusting volume measures using a Herfindahl-Hirschman index for premiums and reserves at a line of business level. However, one misses there a theoretical explanation for the proposed diversification factor. For simplicity, let $V = \sum_{j=1}^{n} V_j$ be the geographical decomposition of the volume measure of a line of business into n geographical regions. Let us assume that diversification can be measured by the *intra-portfolio correlation* coefficient

$$Q = \sum_{i=1}^{n}\sum_{j=1}^{n} \rho_{ij} w_i w_j \in [-1,1], \quad w_i = \frac{V_i}{V}, \tag{2.8}$$

where ρ_{ij} represent the correlation coefficients and w_i the portfolio weights of the non-life risks in the geographical regions. Adjusting for diversification the QIS4 non-life risk capital can be represented as

$$\frac{1}{2}(1+Q) \cdot EC_\alpha[L], \tag{2.9}$$

where $EC_\alpha[L]$ is the original non-life risk capital charge, which does not take diversification into account. If $Q = 1$ (perfect positive dependence between the regions) no reduction for diversification occurs while if $Q = -1$ (perfect negative dependence) the non-life risk capital charge vanishes. If one assumes further a linear dependence structure between perfect dependence and independence such that the correlation coefficients are given by

$$\rho_{ij} = \frac{1}{2} + \frac{1}{2}\delta_{ij}, \quad \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}, \tag{2.10}$$

then one obtains

$$Q = \frac{1}{2}(1+H), \quad H = \sum_{i=1}^{n} w_i^2 , \tag{2.11}$$

where H denotes the *Herfindahl-Hirschman index* (see Hürlimann(2008b) for motivating this choice). In this simple model the non-life risk capital charge reads (QIS4(2008), TS.XIII.B33, p.222)

$$(0.75 + 0.25 \cdot H) \cdot EC_\alpha[L]. \tag{2.12}$$

3. Diversification in a multivariate elliptical model

In general, an adjustment for diversification will be based on the theory of risk measures. Let X be the overall non-life risk per volume unit and let X_j, $j=1,...,n$, be the non-life risks per volume unit in the geographical regions. Then one has the equality $X \cdot V = \sum_{j=1}^{n} X_j \cdot V_j$. Using a positively homogeneous risk measure $\rho(\cdot)$, the non-life risk capital, which has been adjusted for diversification, has the representation

$$EC_\rho(X,V) = \rho(X)\cdot V = DF \cdot \sum_{j=1}^{n} \rho(X_j)\cdot V_j, \tag{3.1}$$

where

$$DF = \frac{\rho(X)\cdot V}{\sum_{j=1}^{n} \rho(X_j)\cdot V_j} \tag{3.2}$$

is the *diversification factor* of the non-life portfolio with respect to the risk measure $\rho(\cdot)$ and $\sum_{j=1}^{n} \rho(X_j)\cdot V_j$ is the non-life risk capital before diversification (sum of the stand-alone non-life risk capitals over the geographical regions). Consider first a class of multivariate distributions of the risk vector $(X_1,...,X_n)$ for which the risk measure $\rho(\cdot)$ is proportional to the standard deviation of the risk. For example, this is the case for the value-at-risk and the conditional value-at-risk measures for the class of multivariate elliptical distributions (e.g. Landsman and Valdez(2003), Dhaene et al.(2008)), which contains the ubiquitous multivariate normal distributions. In this situation one has

$$DF = \frac{\sigma \cdot V}{\sum_{j=1}^{n} \sigma_j \cdot V_j} \tag{3.3}$$

with σ, σ_j, $j=1,...,n$ the standard deviations of X, X_j, $j=1,...,n$. Clearly one has

$$\sigma \cdot V = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} \rho_{ij}(\sigma_i V_i)(\sigma_j V_j)}, \tag{3.4}$$

with ρ_{ij} the correlation coefficients of the non-life risks in the geographical regions. For illustration and comparison purposes assume (2.10). Then one obtains

$$DF = DF(H(\sigma)) = \sqrt{\tfrac{1}{2}(1+H(\sigma))}, \tag{3.5}$$

with

$$H(\sigma) = \frac{\sum_j (w_j\sigma_j)^2}{(\sum_j w_j\sigma_j)^2} \tag{3.6}$$

a *volatility weighted Herfindahl-Hirschman index*. A maximum diversification effect is obtained for a minimum diversification factor or equivalently a minimum value of $H(\sigma)$ subject to the constraint $\sum_{j=1}^{n} w_j = 1$. Applying the Lagrange multiplier method one sees that a solution of this optimization problem solves the equations

$$\sigma_k \cdot \left(\frac{w_k \sigma_k}{\sum_j w_j \sigma_j} - H(\sigma) \right) = \frac{\lambda}{2}, \quad k = 1,...,n, \quad \sum_{j=1}^{n} w_j = 1, \tag{3.7}$$

for some constant λ. The obvious solution with $\lambda = 0$ is

$$w_k = \sigma_k^{-1} \cdot (\sum_{j=1}^{n} \sigma_j^{-1})^{-1}, \quad k = 1,...,n. \tag{3.8}$$

In this situation the minimum diversification factor for n regions equals

$$DF_{\min}^{n} = DF(H(\sigma) = \tfrac{1}{n}) = \sqrt{\tfrac{1}{2}(1 + \tfrac{1}{n})}. \tag{3.9}$$

Asymptotically one obtains the limiting minimum value

$$DF_{\min} = \lim_{n \to \infty} DF_{\min}^{n} = \tfrac{\sqrt{2}}{2}. \tag{3.10}$$

Compared to the QIS4 limiting minimum value of 0.75 in (2.12), the multivariate elliptical model allows for an additional diversification reduction of maximum magnitude 4.29%.

4. <u>Diversification in a multivariate log-normal model</u>

Unfortunately, the simple results of Section 3 do not apply directly to the current Solvency II approach to non-life risk because it relies on log-normal distributions of the risks as seen in Section 2. The portfolio non-life risk per unit of volume, given by $X = \sum_{j=1}^{n} w_j X_j$, is a sum of correlated log-normal random variables, whose distribution does not have an analytical closed-form expression, but can be approximated by means of several methods. In the context of Solvency II we assume that the random vector $(X_1,...,X_n)$ is of the form $(e^{Z_1},...,e^{Z_n})$, where $(Z_1,...,Z_n)$ has a multivariate normal distribution with mean vector $(E[Z_1],...,E[Z_n]) = (-\frac{1}{2}\xi_1^2,...,-\frac{1}{2}\xi_n^2)$, variance vector $(Var[Z_1],...,Var[Z_n]) = (\xi_1^2,...,\xi_n^2)$, and covariance matrix $(Cov[Z_i, Z_j]) = (\theta_{ij}\xi_i\xi_j)$. This assumption is consistent with the requirement $(E[X_1],...,E[X_n]) = (1,...,1)$, that is the expected targets of the combined ratios are one as explained in Section 2. Furthermore, with the variance notation $\sigma_i^2 = Var[X_i]$, $i = 1,...,n$, one has the relationship $\theta_{ij}\xi_i\xi_j = \ln\{1 + \rho_{ij}\sigma_i\sigma_j\}$. For illustration we assume that ρ_{ij} is again specified by (2.10). We discuss two approximation methods.

4.1. Simple log-normal approximation

Firstly and most simply the portfolio combined ratio $X = \sum_{j=1}^{n} w_j X_j$ is approximated by a single log-normal random variable with mean and variance

$$E[X] = \sum_{j=1}^{n} w_j E[X_j] = \sum_{j=1}^{n} w_j = 1 \tag{4.1}$$

$$\sigma^2 = Var[X] = \sum_{j=1}^{n} (w_j \sigma_j)^2 + \sum_{i<j} w_i w_j \sigma_i \sigma_j = \frac{1}{2}(1 + H(\sigma)) \cdot \left(\sum_{j=1}^{n} w_j \sigma_j \right)^2, \tag{4.2}$$

where $H(\sigma)$ is the volatility weighted Herfindahl-Hirschman index defined in (3.6). It is important to mention that this is only a rough log-normal approximation, which can be replaced by a more sophisticated single log-normal approximation if necessary (e.g. Fenton-Wilkinson(1960), Schwartz and Yeh(1982), Beaulieu and Xie(2004), Mehta et al.(2007)). A theoretical justification for the use of such approximations is found in Dufresne(2002). Now, for a minimum capital charge (2.5) or (2.7) under this approximation, one has to minimize (4.2) subject to the constraint $\sum_{j=1}^{n} w_j = 1$. Applying the Lagrange multiplier method one sees that a solution of this optimization problem solves the equations

$$\sigma_k \cdot (\sum_{j=1}^{n} w_j \sigma_j) = \frac{\lambda}{2}, \quad k = 1,...,n, \quad \sum_{j=1}^{n} w_j = 1, \tag{4.3}$$

for some constant λ. This is only possible provided $\sigma_k = \sigma^*, k = 1,...,n,$ that is the volatilities are constant in each geographical region. In this situation $H(\sigma) = H$ coincides with the Herfindahl index (2.11) and a calculation using the relationship (4.2) yields

$$\sigma_k = \frac{\sigma}{\sqrt{\frac{1}{2}(1+H)}}, \ k = 1,...,n. \tag{4.4}$$

The corresponding diversification factor (3.2) reads

$$DF = DF(H) = \frac{\rho_\alpha(\sigma)}{\rho_\alpha(\sigma / \sqrt{\frac{1}{2}(1+H)})}, \tag{4.5}$$

where $\rho_\alpha(\cdot)$ is either (2.5) or (2.7). Its absolute minimum is attained when $H \to 0$ and given by

$$DF_{\min} = \lim_{H \to 0} DF(H) = \frac{\rho_\alpha(\sigma)}{\rho_\alpha(\sqrt{2} \cdot \sigma)}. \tag{4.6}$$

In the current standard Solvency II approach one sets $\alpha = 0.995$ for the VaR measure (2.5) and $\alpha = 0.98675$ for the CVaR measure (2.7) to get approximately $\rho_\alpha(\sigma) \approx 3 \cdot \sigma$ (see also

Table 2.1 in Hürlimann(2008a)). Under this approximation (4.5) reads $DF \approx \sqrt{\frac{1}{2}(1+H)}$ as in Section 3. An exact evaluation of (4.6) yields the following results.

Table 4.1: minimum diversification factor for the simple log-normal approximation

	VaR	DF_min	CVaR	DF_min
confidence level	0.995	0.995	0.98675	0.98675
stdev				
12.0%	2.925	0.673	2.923	0.672
12.5%	2.940	0.672	2.939	0.671
13.0%	2.955	0.670	2.954	0.669
13.5%	2.970	0.669	2.969	0.668
14.0%	2.985	0.668	2.985	0.667
14.5%	3.000	0.667	3.000	0.666
15.0%	3.015	0.666	3.015	0.665
15.5%	3.030	0.665	3.031	0.663
16.0%	3.045	0.663	3.046	0.662
16.5%	3.060	0.662	3.062	0.661
17.0%	3.075	0.661	3.077	0.660

In this table the VaR and the CVaR columns represent the quotients $\rho_\alpha(\sigma)/\sigma$. Compared to the QIS4 limiting minimum value of 0.75 in (2.12), the simple approximation of the multivariate log-normal model allows for an additional diversification reduction of average magnitude 8.3%. In case the volatilities in the geographical regions are not available or difficult to estimate, the assumption of constant volatilities is appropriate and justified by the above minimum property. Alternatively, by given volatility structure $\sigma_k,\ k=1,...,n,$ one can minimize $H(\sigma)$ in (4.2) subject to the constraint $\sum_{j=1}^n w_j = 1$ to get again the optimal weights (3.8). In this situation the diversification factor reads

$$DF_{\min}^n = \frac{\rho_\alpha\left(\sqrt{\frac{1}{2}(1+\frac{1}{n})}\cdot\overline{\sigma}\right)}{\frac{\overline{\sigma}}{n}\cdot\sum_{j=1}^n \frac{\rho_\alpha(\sigma_j)}{\sigma_j}}, \quad \frac{1}{\overline{\sigma}} = \frac{1}{n}\cdot\sum_{j=1}^n \frac{1}{\sigma_j}. \tag{4.7}$$

In the special case of equal volatilities one recovers (4.6) when $n \to \infty$.

4.2. Comonotonic maximum variance approximation

Our second approximation of the sum of correlated log-normal random variables relies on the comonotonic approximation method considered originally in Kaas et al.(2000) and Dhaene et al.(2002). The developments by Vanduffel et al.(2005/2008) suits exactly our needs. Recall that $X = \sum_{j=1}^n w_j e^{Z_j}$, where $(Z_1,...,Z_n)$ satisfies the assumptions at the beginning of this Section. Consider the conditioning random variable Λ, which is defined by

$$\Lambda = \sum_{i=1}^n \gamma_i Z_i \tag{4.8}$$

for some constants γ_i. Following Kaas et al.(2000) one defines a random variable

$$X^{\ell} = E[X|\Lambda] = \sum_{j=1}^{n} w_j \exp\left\{-\tfrac{1}{2}(r_j\xi_j)^2 + r_j\xi_j \tfrac{\Lambda - E[\Lambda]}{\sigma_\Lambda}\right\} \tag{4.9}$$

where $r_j\xi_j\sigma_\Lambda = Cov[Z_j, \Lambda] = \sum_{k=1}^{n} \gamma_k Cov[Z_j, Z_k]$, $j = 1,...,n$. One finds the equality in distribution

$$X^{\ell} =_d \sum_{j=1}^{n} w_j \exp\left\{-\tfrac{1}{2}(r_j\xi_j)^2 + r_j\xi_j \Phi^{-1}(U)\right\}, \tag{4.10}$$

with $\Phi(x)$ the standard normal distribution and U a uniform random variable on $(0,1)$. If all the correlation coefficients r_j defined in (4.9) are non-negative, then X^{ℓ} is a comonotonic sum. In this situation it is well-known that the VaR and CVaR risk measures are determined by (e.g. Vanduffel et al.(2005), Section 2.1)

$$\begin{aligned} VaR_\alpha[X^{\ell}] &= \sum_{j=1}^{n} w_j \exp\left\{-\tfrac{1}{2}(r_j\xi_j)^2 + r_j\xi_j \Phi^{-1}(\alpha)\right\}, \\ CVaR_\alpha[X^{\ell}] &= \frac{1}{1-\alpha} \cdot \sum_{j=1}^{n} w_j \Phi\left(r_j\xi_j - \Phi^{-1}(\alpha)\right). \end{aligned} \tag{4.11}$$

From the definitions in (4.9) one sees that a sufficient condition for $r_j \geq 0$ is that all $\gamma_j \geq 0$ and all $Cov[Z_j, Z_k] \geq 0$. Using Jensen's inequality it can be proved that X^{ℓ} is a convex lower bound of X, a fact written $X^{\ell} \leq_{cx} X$, which means that for any convex function $v(x)$ one has $E[v(X^{\ell})] \leq E[v(X)]$. In Dhaene et al.(2002) a comonotonic convex upper bound, denoted by X^u and such that $X \leq_{cx} X^u$, has also been proposed. In the lognormal context this random variable can be defined by imposing $r_j = 1$ in (4.9). For this upper bound one has

$$X^{u} =_d \sum_{j=1}^{n} w_j \exp\left\{-\tfrac{1}{2}\xi_j^2 + \xi_j \Phi^{-1}(U)\right\}, \tag{4.12}$$

It is easy to see that the VaR and CVaR measures associated to (4.12) correspond to the sum of the stand-alone measures in each geographical region, hence to the valuation before diversification. Since $X^{\ell} \leq_{cx} X \leq_{cx} X^u$ the following relationships hold:

$$E[X^{\ell}] = E[X] = E[X^u] = \sum_{j=1}^{n} w_j = 1, \tag{4.13}$$

$$\begin{aligned} Var[X^{\ell}] &= \sum_{i,j=1}^{n} w_i w_j \left(e^{r_i r_j \xi_i \xi_j} - 1\right) \leq Var[X] = \sum_{i,j=1}^{n} w_i w_j \left(e^{\theta_{ij}\xi_i\xi_j} - 1\right) \\ &\leq Var[X^u] = \sum_{i,j=1}^{n} w_i w_j \left(e^{\xi_i\xi_j} - 1\right). \end{aligned} \tag{4.14}$$

For more details on these results we refer to Kaas et al.(2000) and Dhaene et al.(2002). In view of the inequality (4.14), it is clear that the best comonotonic lower bound approximations of X are the ones for which $Var[X^{\ell}]$ is as close to $Var[X]$ as possible. Vanduffel et al.(2005) maximize the first order approximation of $Var[X^{\ell}]$ obtained by letting $e^{r_i r_j \xi_i \xi_j} - 1 \approx r_i r_j \xi_i \xi_j$ to get the following coefficients in (4.8)

$$\gamma_j = w_j, \quad j = 1,...,n. \tag{4.15}$$

This simple choice is retained here and defines the so-called *comonotonic maximum variance approximation* of X. For approximation purposes we will assume that $\theta_{ij} \approx \rho_{ij}$, where the latter is again specified by (2.10). Then the coefficients r_j in (4.11) are obtained from

$$\begin{gathered}
\sigma_\Lambda^2 = \sum_{i=1}^{n} (w_i \xi_i)^2 + \sum_{i<j} (w_i \xi_i)(w_j \xi_j) = \frac{1}{2}(1 + H(\xi)) \cdot S^2, \\
H(\xi) = \frac{\sum_{j=1}^{n} (w_j \xi_j)^2}{S^2}, \quad S = \sum_{j=1}^{n} w_j \xi_j, \\
r_j = \frac{w_j \xi_j + \frac{1}{2} \sum_{k \neq j} w_k \xi_k}{\sigma_\Lambda} = \frac{\sqrt{2}}{2} \cdot \frac{1 + \frac{w_j \xi_j}{S}}{\sqrt{1 + H(\xi)}}.
\end{gathered} \tag{4.16}$$

It is useful to derive lower and upper bounds to (4.11). For this set $\xi_{\min} = \min_{1 \le j \le n} \xi_j$, $\xi_{\max} = \max_{1 \le j \le n} \xi_j$, and let $\xi_0 = \xi_{\min}$ (lower bound) or $\xi_0 = \xi_{\max}$ (upper bound) in the following. Lower and upper bounds are then obtained from the formula

$$r_j \xi_j = \frac{\sqrt{2}}{2} \cdot \frac{1 + w_j}{\sqrt{1 + H}} \xi_0, \; j = 1,...,n, \quad H = \sum_{j=1}^{n} w_j^2. \tag{4.17}$$

In the special case of equal weights $w_j = \frac{1}{n}$ the corresponding diversification factors read

$$DF^n = \frac{\rho_\alpha(\sqrt{\frac{1}{2}(1 + \frac{1}{n})} \cdot \xi_0)}{\rho_\alpha(\xi_0)}, \tag{4.18}$$

where $\rho_\alpha(\cdot)$ is either (2.5) or (2.7). The absolute minimum of (4.18) is attained when $n \to \infty$ and is given by

$$DF_{\min} = \lim_{n \to \infty} DF^n = \frac{\rho_\alpha(\frac{\sqrt{2}}{2} \cdot \xi_0)}{\rho_\alpha(\xi_0)}. \tag{4.19}$$

With $\xi_0 = \sigma^* = \sqrt{2} \cdot \sigma$ one recovers (4.6) and the numerical results of Table 4.1. We conclude that in the limiting case of minimum diversification the simple log-normal approximation and the comonotonic maximum variance approximation lead up to parameter transformation to the same results.

5. Diversification in a multivariate log-elliptical model

A natural generalization of the multivariate log-normal distribution is the class of multivariate log-elliptical distributions, which has been discussed recently in Dhaene et al.(2008) and Valdez et al.(2009).

In generalization to Section 4, we assume that the random vector $(X_1,...,X_n)$ is of the form $(e^{Z_1},...,e^{Z_n})$, where $(Z_1,...,Z_n)$ has a multivariate elliptical distribution with density generator $g(x)$, mean vector $(E[Z_1],...,E[Z_n]) = (-\ln g(-\xi_1^2),...,-\ln g(-\xi_n^2))$, variance vector $(Var[Z_1],...,Var[Z_n]) = (-2g'(0)\xi_1^2,...,-2g'(0)\xi_n^2)$, and covariance matrix $(Cov[Z_i,Z_j]) = (-2g'(0)\theta_{ij}\xi_i\xi_j)$. This assumption is again consistent with the requirement $(E[X_1],...,E[X_n]) = (e^{E[Z_1]}g(-\xi_1^2),...,e^{E[Z_n]}g(-\xi_n^2)) = (1,...,1)$ of Section 2. Furthermore, with the variance notation $\sigma_i^2 = Var[X_i], i=1,...,n$, one has the relationship

$$1+\rho_{ij}\sigma_i\sigma_j = \frac{g\left(-(\xi_i^2+\xi_j^2+2\theta_{ij}\xi_i\xi_j)\right)}{g(-\xi_i^2)g(-\xi_j^2)}. \tag{5.1}$$

In the log-normal special case one has $g(x)=\exp(-\frac{1}{2}x)$ and (5.1) is equivalent with the relationship $\theta_{ij}\xi_i\xi_j = \ln\{1+\rho_{ij}\sigma_i\sigma_j\}$ of Section 4. In our illustrative examples we assume that $g'(0) = -\frac{1}{2}$, and that ρ_{ij} is again specified by (2.10).

5.1. Simple log-elliptical approximation

In parallel to Section 4.1 the portfolio combined ratio $X = \sum_{j=1}^{n} w_j X_j$ is approximated by a single log-elliptical random variable with mean $E[X]=1$ and variance

$$\sigma^2 = Var[X] = \frac{1}{2}(1+H(\sigma))\cdot\left(\sum_{j=1}^{n} w_j\sigma_j\right)^2, \tag{5.1}$$

where $H(\sigma)$ is defined in (3.6). As in Section 4.1 a minimum capital charge under this approximation is only possible provided $\sigma_k = \sigma^*, k=1,...,n$. In this situation $H(\sigma)=H$ coincides with (2.11). The corresponding diversification factor reads

$$DF = DF(H) = \frac{\rho_\alpha(\sigma)}{\rho_\alpha\left(\sigma/\sqrt{\frac{1}{2}(1+H)}\right)}, \tag{5.2}$$

where $\rho_\alpha(\cdot)$ is either $\rho_\alpha(\sigma) = VaR_\alpha[X]-1$ or $\rho_\alpha(\sigma) = CVaR_\alpha[X]-1$. Its absolute minimum is attained when $H \to 0$ and given by

$$DF_{\min} = \lim_{H\to 0} DF(H) = \frac{\rho_\alpha(\sigma)}{\rho_\alpha(\sqrt{2}\cdot\sigma)}. \tag{5.3}$$

To illustrate consider a multivariate log-Laplace model with density generator $g(x)=(1+\frac{1}{2}x)^{-1}$. Set $\alpha=0.9877$ for the VaR measure and $\alpha=0.96471$ for the CVaR measure to get approximately $\rho_\alpha(\sigma)\approx 3\cdot\sigma$ (choice consistent with QIS4 calibration). An exact evaluation of (5.3) is found in the Table 5.1 and is based on the formulas

$$\begin{aligned} &VaR_\alpha[X]=\left(1-\tfrac{\sqrt{2}}{2}\xi\right)\cdot CVaR_\alpha[X], \quad CVaR_\alpha[X]=\left(1+\tfrac{\sqrt{2}}{2}\xi\right)\cdot[2(1-\alpha)]^{-\frac{\sqrt{2}}{2}\xi}, \\ &\xi=\sqrt{2\sqrt{1+5\sigma^2+4\sigma^4}-2(1+2\sigma^2)}<\sqrt{2}, \end{aligned} \tag{5.4}$$

where the latter expression follows from the general log-elliptical relationship

$$1+\sigma^2=\frac{g(-4\xi^2)}{g(-\xi^2)^2}, \quad -2g'(0)\cdot\xi^2=Var[\ln X] \tag{5.5}$$

by noting that $g(x)=(1+\frac{1}{2}x)^{-1}$ and solving (5.5) for ξ.

<u>Table 5.1</u>: minimum diversification factor for the simple log-Laplace approximation

	VaR	DF_min	CVaR	DF_min
confidence level	0.9877	0.9877	0.96471	0.96471
stdev				
12.0%	2.943	0.682	2.934	0.678
12.5%	2.955	0.681	2.947	0.677
13.0%	2.966	0.681	2.960	0.676
13.5%	2.978	0.681	2.974	0.676
14.0%	2.989	0.681	2.987	0.675
14.5%	3.000	0.680	3.000	0.675
15.0%	3.011	0.680	3.012	0.674
15.5%	3.021	0.680	3.025	0.674
16.0%	3.032	0.680	3.037	0.674
16.5%	3.042	0.680	3.050	0.674
17.0%	3.052	0.680	3.062	0.673

Compared to the log-normal results of Table 4.1, the simple approximation of the multivariate log-Laplace model leads to similar capital charges for significantly lower confidence levels, which are due to the fat tails of this model. The diversification reduction of approximate magnitude 7% compared to QIS4 is a bit less than for the log-normal model. A formula similar to (4.7) can also be derived.

5.2. <u>A Taylor based mean-preserving approximation</u>

Our second approximation of the sum of correlated log-elliptical random variables is based on Valdez et al.(2008). Recall that $X=\sum_{j=1}^{n}w_j e^{Z_j}$, where $(Z_1,...,Z_n)$ satisfies the assumptions at the beginning of this Section. Consider the conditioning random variable Λ, which is defined by

$$\Lambda = \sum_{i=1}^{n} \gamma_i Z_i \tag{5.6}$$

for some constants γ_i. Following Valdez et al.(2008), Section 6, one defines a random variable

$$X^{MP} = \sum_{j=1}^{n} w_j \cdot g\left(-\left(r_j \xi_j\right)^2\right)^{-1} \cdot \exp\left\{r_j \xi_j \tfrac{\Lambda - E[\Lambda]}{\sigma_\Lambda}\right\}, \tag{5.7}$$

where $r_j \xi_j \sigma_\Lambda = Cov\left[Z_j, \Lambda\right] = \sum_{k=1}^{n} \gamma_k Cov\left[Z_j, Z_k\right]$, $j = 1,...,n$. One finds the equality in distribution

$$X^{MP} =_d \sum_{j=1}^{n} w_j \cdot g\left(-\left(r_j \xi_j\right)^2\right)^{-1} \cdot \exp\left(r_j \xi_j F_Z^{-1}(U)\right), \tag{5.8}$$

with $F_Z(x)$ the spherical distribution with density generator $g(x)$ and U a uniform random variable on $(0,1)$. Since $E\left[X^{MP}\right] = E[X]$ the approximation (5.7) is a mean-preserving approximation. Moreover, if $g(x) = e^{-\frac{1}{2}x}$, then (5.8) coincides with the comonotonic log-normal approximation (4.9) (similar to Valdez et al.(2008), Theorem 6.1). The VaR and CVaR risk measures of (5.8) are determined by (e.g. Valdez and Dhaene(2004))

$$\begin{aligned} VaR_\alpha\left[X^{MP}\right] &= \sum_{j=1}^{n} w_j \cdot g\left(-\left(r_j \xi_j\right)^2\right)^{-1} \cdot \exp\left(r_j \xi_j F_Z^{-1}(\alpha)\right), \\ CVaR_\alpha\left[X^{MP}\right] &= \frac{1}{1-\alpha} \cdot \sum_{j=1}^{n} w_j \cdot g\left(-\left(r_j \xi_j\right)^2\right)^{-1} \cdot \overline{F}_{Z_j^*}\left(F_Z^{-1}(\alpha)\right), \end{aligned} \tag{5.9}$$

where Z_j^* is the Escher transform of Z with parameter $r_j \xi_j$, whose density is defined by

$$f_{Z_j^*}(x) = g\left(-\left(r_j \xi_j\right)^2\right)^{-1} \cdot \exp\left(r_j \xi_j x\right) \cdot f_Z(x). \tag{5.10}$$

Valdez et al.(2008) have suggested to choose the coefficients in (5.6) such that Λ and X are "as alike as" possible, which results in the so-called *Taylor based mean-preserving approximation* (see also Vanduffel et al.(2008)) with coefficients (5.6) given by

$$\gamma_j = g\left(-\xi_j^2\right) \cdot w_j, \quad j = 1,...,n. \tag{5.11}$$

For approximation purposes we will as in Section 4.2 assume that $\theta_{ij} \approx \rho_{ij}$, where the latter is specified by (2.10). Then the coefficients r_j in (5.9) are obtained from

$$\sigma_\Lambda^2 = \sum_{i=1}^{n} (w_i g(-\xi_j^2)\xi_i)^2 + \sum_{i<j} (w_i g(-\xi_j^2)\xi_i)(w_j g(-\xi_j^2)\xi_j) = \frac{1}{2}(1+H(\xi)) \cdot S^2,$$

$$H(\xi) = \frac{\sum_{j=1}^{n} (w_j g(-\xi_j^2)\xi_j)^2}{S^2}, \quad S = \sum_{j=1}^{n} w_j g(-\xi_j^2)\xi_j, \tag{5.12}$$

$$r_j = \frac{w_j g(-\xi_j^2)\xi_j + \frac{1}{2}\sum_{k \neq j} w_k g(-\xi_k^2)\xi_k}{\sigma_\Lambda} = \frac{\sqrt{2}}{2} \cdot \frac{1 + \frac{w_j g(-\xi_j^2)\xi_j}{S}}{\sqrt{1+H(\xi)}}.$$

It is useful to derive lower and upper bounds to (5.9). For this set $\xi_{\min} = \min_{1 \le j \le n} \xi_j$, $\xi_{\max} = \max_{1 \le j \le n} \xi_j$, and let $\xi_0 = \xi_{\min}$ (lower bound) or $\xi_0 = \xi_{\max}$ (upper bound) in the following. Lower and upper bounds are then obtained from the formula

$$r_j \xi_j = \frac{\sqrt{2}}{2} \cdot \frac{1+w_j}{\sqrt{1+H}} g(-\xi_0^2)\xi_0, \ j = 1,...,n, \quad H = \sum_{j=1}^{n} w_j^2. \tag{5.13}$$

In the special case of equal weights $w_j = \frac{1}{n}$ the corresponding diversification factors read

$$DF^n = \frac{\rho_\alpha(\sqrt{\frac{1}{2}(1+\frac{1}{n})} \cdot g(-\xi_0^2)\xi_0)}{\rho_\alpha(g(-\xi_0^2)\xi_0)}, \tag{5.14}$$

where $\rho_\alpha(\cdot)$ is either $\rho_\alpha(\sigma) = VaR_\alpha[X] - 1$ or $\rho_\alpha(\sigma) = CVaR_\alpha[X] - 1$. The absolute minimum of (5.14) is attained when $n \to \infty$ and is given by

$$DF_{\min} = \lim_{n \to \infty} DF^n = \frac{\rho_\alpha(\frac{\sqrt{2}}{2} \cdot g(-\xi_0^2)\xi_0)}{\rho_\alpha(g(-\xi_0^2)\xi_0)}. \tag{5.15}$$

With $g(-\xi_0^2)\xi_0 = \sigma^*$ one recovers (5.3) and the numerical results of Table 5.1 for the multivariate log-Laplace model. We conclude that in the limiting case of minimum diversification the simple log-elliptical approximation and the Taylor based mean-preserving approximation lead up to parameter transformation to the same results.

6. Application to the current Solvency II standard approach

It appears instructive to consider the impact of our findings on the current Solvency II standard approach. We give a numerical example, which compares the current QIS4 specification with the new approach based on the common assumption of log-normally distributed non-life risks. For illustration purposes it suffices to restrict the analysis to the simple log-normal approximation of Section 4.1. We suppose that the volatilities in the geographical regions of a line of business are unknown, and assume therefore that they are constant in each line of business (as motivated in Section 4.1). For the determination of the solvency capital requirement (SCR) for the combined premium and reserve risk the following data is required:

m	: number of lines of business
V_ℓ	: volume measure of the line of business $\ell \in \{1,...,m\}$
σ_ℓ	: volatility measure (standard deviation) of the line of business ℓ
H_ℓ	: Herfindahl index of the line of business ℓ
$C = (\rho_{k\ell})$	: correlation matrix between the lines of business $k, \ell \in \{1,...,m\}$

Let $V = \sum_{\ell=1}^{m} V_\ell$ be the overall volume measure and consider the volume weights $w_\ell = V_\ell / V$, $\ell \in \{1,...,m\}$, and the vector of weighted volatilities $\sigma_w = (w_1\sigma_1,...,w_m\sigma_m)$. Then, the overall standard deviation σ is obtained from the equation $\sigma^2 = \sigma_w^T \cdot C \cdot \sigma_w$. Without geographical diversification the capital requirement for premium and reserve risk at the confidence level $\alpha = 99.5\%$ is given by (2.4), that is

$$SCR_{PR} = \rho_\alpha(\sigma) \cdot V \,. \tag{6.1}$$

To take geographical diversification into account according to QIS4, one considers the geographically diversified volume measures

$$V_\ell^D = (0.75 + 0.25 \cdot H_\ell) \cdot V_\ell, \quad \ell \in \{1,...,m\}. \tag{6.2}$$

Let $V^D = \sum_{\ell=1}^{m} V_\ell^D$ be the overall diversified volume measure and consider the diversified volume weights $w_\ell^D = V_\ell^D / V^D$, $\ell \in \{1,...,m\}$, and the vector of diversified weighted volatilities $\sigma_w^D = (w_1^D\sigma_1,...,w_m^D\sigma_m)$. Then, the overall diversified standard deviation σ^D is obtained from the equation $(\sigma^D)^2 = (\sigma_w^D)^T \cdot C \cdot \sigma_w^D$. With geographical diversification the capital requirement for premium and reserve risk at the confidence level $\alpha = 99.5\%$ is now

$$SCR_{PR}^D = \rho_\alpha(\sigma^D) \cdot V^D \,. \tag{6.3}$$

Alternatively, according to the simple log-normal approximation of Section 4.1, one considers the geographically diversified volume measures, which are consistent with (4.5) and defined by

$$\tilde{V}_\ell^D = \frac{\rho_\alpha(\sigma_\ell)}{\rho_\alpha(\sigma_\ell / \sqrt{\tfrac{1}{2}(1+H_\ell)})} \cdot V_\ell, \quad \ell \in \{1,...,m\}. \tag{6.4}$$

Let $\tilde{V}^D = \sum_{\ell=1}^{m} \tilde{V}_\ell^D$ be the corresponding overall diversified volume measure and consider the diversified volume weights $\tilde{w}_\ell^D = \tilde{V}_\ell^D / \tilde{V}^D$, $\ell \in \{1,...,m\}$, and the vector of diversified weighted volatilities $\tilde{\sigma}_w^D = (\tilde{w}_1^D\sigma_1,...,\tilde{w}_m^D\sigma_m)$. The corresponding overall diversified standard deviation $\tilde{\sigma}^D$ is obtained from the equation $(\tilde{\sigma}^D)^2 = (\tilde{\sigma}_w^D)^T \cdot C \cdot \tilde{\sigma}_w^D$. With geographical diversification the alternative simple log-normal capital requirement for premium and reserve risk at the confidence level $\alpha = 99.5\%$ is given by

$$\widetilde{SCR}_{PR}^{D} = \rho_\alpha\left(\tilde{\sigma}^D\right)\cdot \tilde{V}^D. \tag{6.3}$$

The next table illustrates at two single examples the numerical impact of the new approach under varying levels of geographical diversification as measured by the Herfindahl indices. We suppose that there are $m = 5$ lines of business with the following correlation matrix

$$C = (\rho_{k\ell}) = \begin{pmatrix} 1 & 0.5 & 0.5 & 0.25 & 0.25 \\ 0.5 & 1 & 0.25 & 0.25 & 0.5 \\ 0.5 & 0.25 & 1 & 0.5 & 0.25 \\ 0.25 & 0.25 & 0.5 & 1 & 0.5 \\ 0.25 & 0.5 & 0.25 & 0.5 & 1 \end{pmatrix} \tag{6.4}$$

Table 6.1: QIS4 geographical diversification versus simple log-normal approximation

	overall	lines of business				
volumes	1000	400	250	200	100	50
standard deviations (std)	14.5%	12%	20%	25%	30%	50%
SCR (without Diversification)	**435.6**					
Example 1						
Herfindahl indices		0.25	0.5	0.6	0.75	1
QIS4 diversified volumes	867.5	325	218.75	180	93.75	50
QIS4 diversified overall std	14.9%					
QIS4 SCR (with Diversification)	**387.8**					
alternative diversified volumes	832.7	306.26	210.29	174.21	91.90	50
alternative diversified overall std	14.9%					
alternative SCR (with Diversification)	**375.1**					
Example 2						
Herfindahl indices		0.1	0.2	0.3	0.4	0.5
QIS4 diversified volumes	803.75	310	200	165	85	43.75
QIS4 diversified overall std	14.7%					
QIS4 SCR (with Diversification)	**355.6**					
alternative diversified volumes	741.75	284.45	183.45	152.92	79.67	41.26
alternative diversified overall std	14.8%					
alternative SCR (with Diversification)	**329.3**					

In example 1 the diversification effect equals 11% of the SCR without diversification under the QIS4 approach. Under the alternative approach this effect increases to 13.9%. In the more diversified example 2 the diversification effect increases from 18.4% to 24.4%. Since the line of business diversification factors satisfy the approximations $DF_\ell \approx \sqrt{\tfrac{1}{2}(1+H_\ell)}$ and in virtue of the inequalities

$$\sqrt{\tfrac{1}{2}(1+H)} \leq 0.75 + 0.25\cdot H\,, \tag{6.4}$$

we expect that the diversification effect always increases from the QIS4 approach to the alternative approach, which implies a release of required risk capital.

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