HEDGING METHODOLOGIES IN EQUITY-LINKED LIFE INSURANCE

Alexander Melnikov
University of Alberta, Edmonton
e-mail: **melnikov@ualberta.ca**

1. Formulation of the Problem and Introductory Remarks.

The contracts we are going to study have two types of uncertainty:

- Uncertainty in the framework of given financial market (market risk),
- Mortality of insured.

These types of uncertainty are weakly correlated, and in our setting we shall model them exploiting two different probability spaces:

$$(\Omega, F, P) \text{ and } (\tilde{\Omega}, \tilde{F}, \tilde{P}).$$

Combining both models, we get the product space

$$\left(\Omega \times \tilde{\Omega}, F \times \tilde{F}, P \times \tilde{P}\right).$$

Financial Market is represented by a pair of assets, non-risky B and risky S, that are identified by their prices (B_t) and (S_t) as stochastic processes on (Ω, F, P).

Assuming for simplicity $B_t = 1$, we describe a *filtration*

$$\mathbf{F} = \left(F_t = \sigma\{S_0, ..., S_t\}\right)_{t \geq 0}$$

as the available information about prices at time t.

To give a full description of the market, we should introduce a variety of admissible operations with the basic assets, B and S.

Define a 2-dimensional stochastic process $\pi_t = (\beta_t, \gamma_t)$ (usually predictable) adapted to $\mathbf{F}$, as a *portfolio*, or a *strategy* with the value (capital)

$$X_t^{\pi} = \beta_t B_t + \gamma_t S_t.$$

If X_t^π follows the equation

$$dX_t^\pi = \beta_t dB_t + \gamma_t dS_t,$$

then we call π *self-financing*.

Only portfolios with non-negative capital are *admissible* in the framework of the market.

Any financial contract is identified with its potential liability (payoff) H, exercised at the end of a contract period $[0,T]$. In fact, H is a F_T-measurable random variable. Any such nonnegative random variable H is called a *contingent claim*. In financial economics, a contingent claim is called a *derivative security* of the European type.

Remark: A larger class of derivative securities, the American type, is not considered here.

The *main problem* is to find the current price of a given contingent claim during a contract period $[0,T]$ in order to manage the risk in the framework of the contract.

An *appropriate solution* is found by *hedging the contingent claims:* to construct an admissible strategy π^* so that

$$X_T^{\pi^*} \textit{ is close enough to } H$$

in some probabilistic sense.

Hedging gives us a possibility to determine the current price $\mathbf{C}_t$ at time $t \le T$ as the current capital $\mathbf{C}_t = X_t^{\pi^*}$ of the hedging portfolio π^*.

In particular,

$$\mathbf{C} = X_0^{\pi^*}$$

represents the *initial price* of the claim.

Types of hedging

1. *Perfect* hedging:

$$P\left\{X_T^{\pi*} \geq H\right\} = 1.$$

2. *Mean-variance* hedging:

$$E\left(X_T^{\pi*} - H\right)^2 \text{ is minimal.}$$

3. *Efficient* hedging:

$$E\left\{l\left(H - X_T^{\pi*}\right)^+\right\} \text{ is minimal,}$$

where l is a loss function.

In particular, for *quantile* hedging, we have

$$l(x) = I_{(0,\infty)}(x) \text{ and}$$

$$P\left\{X_T^{\pi*} \geq H\right\} \text{ is maximal.}$$

How to calculate the price $\mathbf{C}$?

To answer this question, we use the martingale characterization of *arbitrage* and *completeness* of a financial market (in terms of existence and uniqueness of *risk-neutral* or *martingale* measures) given by the first and the second fundamental theorems in financial mathematics.

So, we reduce the method of risk-neutral valuation of the contingent claim H to finding the price $\mathbf{C}_t$ as

$$\mathbf{C}_t = E^*(H \mid F_t),$$

where E^* is an expectation w.r. to a single martingale measure P^* (complete market) or

$$\mathbf{C}_t = \sup_{P^*} E^*(H \mid F_t)$$

(incomplete market).

Life insurance contracts based on risky assets

The source of randomness in financial contracts is the evolution of stock prices. The source of randomness in life insurance is the mortality of clients. Denote $T(x)$ a random variable representing the *future lifetime* of a client of age x. $T(x)$ is given on $(\tilde{\Omega}, \tilde{F}, \tilde{P})$.

An insurance company can issue a *mixed* contract for the period $[0, T]$, where the payoff function is a function of stock prices $S_0, ..., S_T$ and $T(x)$.

This type of contract is called *equity-linked life insurance contract*.

We will consider only *pure endowment* contracts with the following structure of payoffs:

$$H(T(x)) = H \cdot I_{\{T(x) > T\}}.$$

Remarks: (1) *Traditional* life insurance deals with the deterministic payoff $H = K = const$. The price of the contract is equal to $K \cdot {}_T p_x$, where ${}_T p_x = \tilde{P}\{T(x) > T\}$ is a *survival probability* of the insured of age x.

(2) *Pure endowment with a fixed guarantee*:

$$H(T(x)) = \max\{S_T, K\} \cdot I_{\{T(x)>T\}},$$

where S is a risky asset, K is a guarantee.

(3) *Pure endowment with a flexible guarantee*:

$$H(T(x)) = \max\left\{S_T^1, S_T^2\right\} \cdot I_{\{T(x)>T\}},$$

where S_T^1 is the risky asset, S_T^2 is the flexible (may be also a risky asset) guarantee.

Assume that S in (2) follows geometric Brownian Motion (Black-Scholes model):

$$S_t = S_0 \exp\left\{\left(\mu - \frac{\sigma}{2}\right)^2 t + \sigma W_t\right\}, \; t \le T,$$

where W is a Wiener process, μ is a rate of return on stock S, and σ is the volatility.

A Brief History

We note the papers by Brennan and Schwartz (1976,1979), Boyle and Schwartz (1977), which recognized a close connection of equity-linked life insurance with the option pricing theory (Black, Scholes, and Merton (1973)). They found that the payoff from equity-linked life insurance contract at expiration is identical to the payoff from a European call option plus some guaranteed amount. Hence, the appearance of Black-Scholes formula in such pricing is quite natural in the construction of the initial price and the value of the portfolio:

$$
\begin{aligned}
{}_T U_x &= {}_T p_x \left[S_0 \Phi(d_+) + K\Phi(-d_-) \right] \\
&= {}_T p_x K + {}_T p_x \left[S_0 \Phi(d_+) - K\Phi(-d_-) \right],
\end{aligned}
$$

$$
d_\pm = \frac{\ln \frac{S_0}{K} \pm \frac{\sigma^2}{2} T}{\sigma \sqrt{T}}, \quad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy .
$$

Further developments: Delbaen (1986), Bacinello and Ortu (1993), Aase and Persson (1994).

The important step was done by Moeller (1998, 2001, 2002) who applied the mean-variance hedging technique. He obtained the following result:

$$U(T, l_x) = l_x \cdot {}_T p_x \left[S_0 \Phi(d_+) + K\Phi(-d_-) \right]$$

$$\text{and } \gamma_t^* = (l_x - N_{t-})_{T-t}\, p_{x+t} F'_x(t, S_t)$$

give the initial price and optimal (in the mean-variance sense) hedging strategy for the contract of pure endowment with a guarantee K for a group of clients of age x, where l_x is the size of the group of the insured with the remaining life times $T_1(x),\ldots,T_{l_x}(x)$,

$$N_t = \sum_{i=1}^{l_x} I_{\{T_i(x) \le t\}}, \; {}_{T-t}p_{x+t} = \tilde{P}(T_i(x+t) > T-t).$$

$F(t,x)$ satisfies the Black-Scholes fundamental equation $(r = 0)$

$$F_t'(t,x)+\frac{1}{2}\sigma^2x^2F_{xx}''(t,x)=0$$

with the boundary condition

$$F(T,x)=\max\{x,K\}.$$

Remark: Our main focus is on calculating the premium ${}_TU_x$ for a single contract. Since for a group of size l_x we get

$$U(T,l_x)=l_x\cdot{}_TU_x.$$

Also, for every fixed $t\le T$ we can repeat the logical steps for the interval $[t,T]$ and find the corresponding premiums as a product

$$(l_x-N_t)\cdot{}_{T-t}U_{x+t}.$$

2.Conditioned Contingent Claims in a semimartingale setting.

Assume that the risky asset S is a semimartingale $S=(S_t,F_t)_{t\le T}$ on $(\Omega, F, \mathbf{F}, P)$.

A self-financing admissible strategy $\pi=(\beta,\gamma)$

has a capital $X_t^{\pi}=X_0^{\pi}+\int_0^t \gamma_u dS_u \ge 0$.

$M(S,P)$ is a set of martingale measures of this market. Firstly, we assume that $M(S,P)=\{P^*\}$.

Consider a nonnegative contingent claim H which is F_t-measurable random variable with $E^* H<\infty$.

According to the risk-neutral valuation methodology,

$$\mathbf{C}_t(H)=E^*(H\,|\,F_t)=\mathbf{C}_0(H)+\int_0^t \gamma_u^H dS_u \ge 0,$$

$$\mathbf{C}_0(H)=E^* H,$$

$\pi^H=\left(\beta^H,\gamma^H\right)$ is a replicating portfolio (perfect hedge).

Consider a nonnegative random variable τ defined on another probability space $\left(\tilde{\Omega}, \tilde{F}, \tilde{P}\right)$ and determine a *conditioned contingent claim*

$$H(\tau) = H \cdot I_{\{\tau > T\}}$$

on the product space $\left(\Omega \times \tilde{\Omega}, F \times \tilde{F}, P \times \tilde{P}\right)$.

To price $H(\tau)$ we consider the value

$$\mathbf{C}(\tau) = E^* \times \tilde{E}(H(\tau)) = E^* H \cdot \tilde{E} I_{\{\tau > T\}}$$
$$= \mathbf{C}_0(H) \cdot p(T) < \mathbf{C}_0(H)$$

as a bound for the initial price of $H(\tau)$ and define the *set of successful hedging*

$$A(X_0^\pi, \pi) = \{\omega : X_T^\pi(X_0^\pi) \geq H\}.$$

If $\pi = \pi^H$ then $P(A(\mathbf{C}_0(H), \pi)) = 1$. We should consider a restricted set of strategies with the initial capital $X_0^\pi \leq X_0 = \mathbf{C}_0(H) \cdot p(T)$. Thus, we cannot provide the above equality.

To price a contingent claim $H(\tau)$, we should consider the following *extreme problem*:

find an admissible strategy π^* such that

$$P(X_T^{\pi^*}(\mathbf{C}(\tau)) \geq H) = \max_{\pi} P(A(X_0^{\pi}, \pi))$$

under the restriction $X_0^{\pi} \leq X_0 = \mathbf{C}(\tau) < \mathbf{C}_0(H)$.

This is exactly the *problem of quantile hedging* (Follmer and Leukert (1999, 2000)). According to this theory, the *optimal* strategy $\pi^* = \left(\beta^*, \gamma^*\right)$ is a perfect hedge for the claim $H_{A^*} = H \cdot I_{A^*}$, where the set $A^* \in F_T$ has a maximal P-probability with $E^* HI_A \leq X_0$. This set is called a *maximal successful hedging set* .

Remarks: In case of an incomplete market, the bound for the initial capital is given by

$$\mathbf{C}_0^*(\tau) = \left(\sup_{P' \in M(S,P)} E'H \right) \cdot p(T) = \mathbf{C}_0^*(H) \cdot p(T).$$

3. Pricing of contracts with a fixed guarantee.

Consider Black-Scholes model for the risky asset

$$S_t = S_0 \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right\}$$

$$\text{or} \quad dS_t = S_t(\mu dt + \sigma dW_t)$$

with the martingale measure

$$P^* : \; Z_T = \frac{dP^*}{dP} = \exp\left\{-\frac{\mu}{\sigma}W_T - \frac{1}{2}\left(\frac{\mu}{\sigma}\right)^2 T\right\}$$

and $W_t^* = W_t + \frac{\mu}{\sigma}t$, the new Wiener process with respect to P^*.

A pure endowment contract with a fixed guarantee is identified with the following contingent claim:

$$H(T(x)) = H \cdot I_{\{T(x)>T\}} = \max(S_T, K) \cdot I_{\{T(x)>T\}} =$$

$$= K \cdot I_{\{T(x)>T\}} + (S_T - K)^+ \cdot I_{\{T(x)>T\}}.$$

The initial price ${}_T U_x$ for this contract can be calculated as

$$ {}_TU_x = \mathbf{C}_0(T(x)) = {}_Tp_x \cdot K + {}_Tp_x \cdot E^*(S_T - K)^+ = $$
$$ = {}_Tp_x \cdot K + {}_Tp_x \cdot (S_0\Phi(d_+) - K\Phi(d_-)). $$

We can consider the quantity

$$ {}_TU_x - {}_Tp_x \cdot K = \mathbf{C}_0(T(x)) = {}_Tp_x \cdot E^*(S_T - K)^+, $$

as the bound of the initial capital available for a call option. Applying the methodology of quantile hedging, we get

$$ {}_Tp_x \cdot E^*(S_T - K)^+ = E^*(S_T - K)^+ \cdot I_{A^*}, $$

or the following convenient form for further actuarial analysis of the contract:

$$ {}_Tp_x = \frac{E^*(S_T - K)^+ \cdot I_{A^*}}{E^*(S_T - K)^+}. $$

The maximal successful hedging set A^* has a special structure:

$$ A^* = \left\{ Z_T^{-1} > a(S_T - K)^+ \right\} = \left\{ S_T^{\mu/\sigma^2} > a_1(S_T - K)^+ \right\}. $$

There are two cases when the equation

$$x^{\mu/\sigma^2} = a_1(x-K)^+$$

has one or two solutions. Case $\mu \le \sigma^2$ is reduced to

$A^* = \{S_T \le c\} = \{W_T^* \le b\}$, and

$$E^*(S_T - K)^+ I_{A^*}$$

$$= S_0\Phi(d_+) - K\Phi(d_-) - \left[S_0\Phi\left(\frac{-b+\sigma T}{\sqrt{T}}\right) - K\Phi\left(\frac{-b}{\sqrt{T}}\right)\right].$$

Finally,

$${}_T p_x = 1 - \frac{S_0\Phi\left(\frac{-b+\sigma T}{\sqrt{T}}\right) - K\Phi\left(\frac{-b}{\sqrt{T}}\right)}{S_0\Phi(d_+) - K\Phi(d_-)}.$$

A constant b (or c) can be found from given probability of successful hedging: an insurance company can be agreed with a certain risk level $\varepsilon \in (0,1)$ such that $1-\varepsilon = P(A^*)$.

Due to the structure of A^*, we can find

$$P\left(A^*\right) = \Phi\left(\frac{b - \frac{\mu}{\sigma}T}{\sqrt{T}}\right)$$

and therefore $b = \sqrt{T}\Phi^{-1}(1-\varepsilon) + \frac{\mu}{\sigma}T$.

Illustrative Numerical Example

Let's fix a risk level $\varepsilon = 0.01$ and specify other parameters of the model and the contract:

$\mu = 0.08; \sigma = 0.3; S_0 = 100; K = 110; T = 1, 3, 5$ years.

We can find that

${}_1p_x = 0.930095;\ {}_3p_x = 0.94826;\ \ {}_5p_x = 0.955106.$

Using Life Tables (Bowers et al, 1997), we can reconstruct the appropriate age of the insured:

$x \geq 78; x \geq 62; \quad x \geq 53$ years.

Black-Scholes prices : 8.141; 16.876; 22.849.

Quantile prices C_ε: for $\varepsilon = 0.01$ are 7.571; 16.003; 22.849 (5-7% lower); for $\varepsilon = 0.03$ are 6.653; 14.514; 20.033 (12-18% lower).

Remark. Consider *cumulative claims*

$l_{x+T}(S_T - K)^+$, where l_{x+T} is the number of insureds at time T from the group of size l_x.

The terminal capital of a quantile hedge $\pi = \pi_\varepsilon$ of the risk level ε satisfies to

$$P(X^\pi{}_T \geq (S_T - K)^+) = 1 - \varepsilon.$$

The maximal set of successful hedging is invariant w.r.to multiplication by a positive constant δ. Hence,

$\pi = \pi_\varepsilon$ represents a quantile hedge for the claim $\delta(S_T - K)^+$ with the initial price δC_ε.

Take $\delta = \dfrac{n_\alpha}{l_x}$, where n_α is defined from the equality

$$P(n_\alpha \geq l_{x+T}) = 1 - \alpha.$$

Parameter $\alpha \in (0,1)$ characterizes the level of *mortality risk*, and this probability is calculated with the help of Binomial Distribution with parameter ${}_T p_x$. Independence l_{x+T} and the market implies

$$P(l_x X_T{}^{\pi} \geq l_{x+T}(S_T - K)^{+}) \geq P(X_T{}^{\pi} \geq \frac{n_\alpha}{l_x}(S_T - K)^{+})P(n_\alpha \geq l_{x+T})$$

$$\geq (1-\varepsilon)(1-\alpha) \geq 1-(\varepsilon+\alpha).$$

So, with the help of strategy $\pi = \pi_{\varepsilon}$ and the initial

price $C_{\varepsilon,\alpha} = \dfrac{n_\alpha}{l_x} C_{\varepsilon}$ one can hedge the given

cumulative claim with the probability $1-(\varepsilon+\alpha)$.

For risk levels $\varepsilon = 0.03$, $\alpha = 0.02$, $T = 1,3,5$, and

$l_x = 100$ we have: $n_\alpha = 89,93,94$ and

$C_{\varepsilon,\alpha} = 5.921;13.498;18.831.$

Therefore, under the risk level at 5%, the initial contract price can be reduced by 18-28%.

Case $\mu > \sigma^2$ leads to two constants $c_1 < c_2$ (or $b_1 < b_2$). The structure of a successful set will be as follows

$$A^{*} = \{W_T^{*} \leq b_1\} \cup \{W_T^{*} > b_2\}$$

and we can do the same as in the previous case.

4. Contracts with flexible guarantees.

The contract under consideration here has the structure

$$H(T(x)) = \max\left\{S_T^1, S_T^2\right\} \cdot I_{\{T(x)>T\}},$$

where S_t^i, $i = 1,2,$ follows the equation

$$dS_t^i = S_t^i(\mu_i dt + \sigma_i dW_t).$$

The first asset S^1 is supposed to be more risky than the second one S^2. Hence, we assume that $\sigma_1 > \sigma_2$ and S^2 plays the role of the flexible guarantee.

The market modelled in this case can be identified with a Black-Scholes model for the asset S^1 because S^2 can be expressed as a power function of S^1.

Hence, we can use a standard martingale measure P^* with the density

$$Z_T = \frac{dP^*}{dP} = \exp\left\{-\frac{\mu_1}{\sigma_1}W_T - \frac{1}{2}\left(\frac{\mu_1}{\sigma_1}\right)^2 T\right\}.$$

We can reduce the problem to the equality

$${}_T p_x = \frac{E^*(S_T^1 - S_T^2)^+ \cdot I_{A^*}}{E^*(S_T^1 - S_T^2)^+}.$$

Assume that $0 < \sigma_1 - \sigma_2 << \sigma_1$ and σ_2.

The structure of the successful hedging set A^* is

$$\left\{Z_T^{-1} > a(S_T^1 - S_T^2)^+\right\} = \left\{\frac{1}{Z_T S_T^2} > a(Y_T - 1)^+\right\}, Y_T = \frac{S_T^1}{S_T^2}.$$

We represent as $Z_T S_T^2 = Y_T^\alpha \cdot const$, and find

the characteristic equation defining A^* as

$$x^{-\alpha} = const \cdot a(x-1)^+,$$

where

$$-\alpha = \frac{\sigma_2 + (\sigma_1 - \sigma_2)\sigma_2}{\sigma_2 - (\sigma_1 - \sigma_2)^2 \sigma_2} \cong 1.$$

Therefore, we can replace the equation above by

$$x = const \cdot a(x-1)^{+}$$

and find its unique solution $c > 1$.

So, we arrive to the equality

$${}_T p_x = \frac{E^*(S_T^1 - S_T^2)^+ \cdot I_{\{Y_T \le c\}}}{E^*(S_T^1 - S_T^2)^+}$$

$$= 1 - \frac{S_0^1 \Phi\left(b_+\left(S_0^1, c\tilde{S}_0^2, T\right)\right) - \tilde{S}_0^2 \Phi\left(b_-\left(S_0^1, c\tilde{S}_0^2, T\right)\right)}{S_0^1 \Phi\left(b_+\left(S_0^1, \tilde{S}_0^2, T\right)\right) - \tilde{S}_0^2 \Phi\left(b_-\left(S_0^1, \tilde{S}_0^2, T\right)\right)},$$

where

$$\tilde{S}_0^2 = E^* S_T^2 = S_0^2 \exp\left\{\frac{\sigma_1\mu_2 - \sigma_2\mu_1}{\sigma_1} T\right\},$$

$$b_\pm\left(S^1, S^2, T\right) = \frac{\ln \frac{S^1}{S^2} \pm (\sigma_1 - \sigma_2)^2 \frac{T}{2}}{(\sigma_1 - \sigma_2)\sqrt{T}}.$$

Both, the numerator and the denominator are variants of *Margrabe's formula*.

We can find c from the given level of risk ε:

$$1 - \varepsilon = P\left(A_{a_\varepsilon^*}^*\right) = P\left(Y_T \le c\left(a_\varepsilon^*\right)\right) = \Phi_{\mu,\sigma^2}\left(\ln c\left(a_\varepsilon^*\right)\right).$$

5. Numerical Example.

Consider the financial indices the Russell 2000 (RUT-I) and the Dow Jones Industrial Average (DJIA) as risky assets S^1 and S^2.

We estimate (μ_1, σ_1) and (μ_2, σ_2) for these indices empirically, using daily observations of prices from August 1, 1997, to July 31, 2003:

$$\mu_1 = 0.0481, \quad \sigma_1 = 0.2232,$$
$$\mu_2 = 0.0417, \quad \sigma_2 = 0.2089.$$

The initial prices of these indices are 414.21 and 8194.04. Therefore, we use $\frac{8194.04}{414.21} \cdot S_t^1$ as the value of the first asset to make initial values of both assets the same $S_0 = 8194.04$.

Utilizing the formulae for c and ${}_T p_x$ with T=1, 3, 5, 10 and $\varepsilon = 0.01,\ 0.025,\ 0.05$, we get the corresponding survival probabilities for the contract with flexible guarantee (Table 1).

We also do the same for the contract with fixed guarantee $K = (1.1) \cdot S_0$ (Table 2).

Using Life Tables (Bowers et al, 1997), we define the corresponding age of the insured for these contracts (Tables 3 and 4).

Whenever the risk that the company will fail to hedge successfully increases, the recommended ages rise as well. This means that the insurance company should compensate by choosing “safer” older clients. We also observed that with longer contract maturities, the company can widen its audience to younger clients.

In both cases (fixed and flexible guarantees) quantile prices (for example, $\varepsilon = 0.025$) are reduced by 7-8% and 9-12%. If the combined ($\alpha = \varepsilon = 0.025$) risk is 5%, the corresponding price reduction will be 12-18%.

Survival Probabilities

Table 1

	$\varepsilon = 0.01$	$\varepsilon = 0.025$	$\varepsilon = 0.05$
T=1	0.9447	0.8774	0.7811
T=3	0.9511	0.8910	0.8041
T=5	0.9549	0.8989	0.8174
T=10	0.9605	0.9108	0.8378

Table 2

	$\varepsilon = 0.01$	$\varepsilon = 0.025$	$\varepsilon = 0.05$
T=1	0.9733	0.9306	0.8585
T=3	0.9700	0.9247	0.8510
T=5	0.9706	0.9266	0.8553
T=10	0.9732	0.9332	0.8679

Age of the Insured

Table 3

	$\varepsilon = 0.01$	$\varepsilon = 0.025$	$\varepsilon = 0.05$
T=1	78	87	94
T=3	61	71	79
T=5	53	63	71
T=10	41	50	58

Table 4

	$\varepsilon = 0.01$	$\varepsilon = 0.025$	$\varepsilon = 0.05$
T=1	68	80	88
T=3	55	67	76
T=5	48	59	68
T=10	36	47	56

6. Further Developments.

We present here how our approach to the pricing of equity-linked life insurance contracts can be extended to other types of efficient hedging and other models of financial market.

6.1. Efficient hedging with power loss function.

$$l(x) = x^p,\ x \ge 0,\ p > 0.$$

Model (with zero interest rate):

$$dS_t^i = S_t^i\left(\mu_i dt + \sigma_i dW_t\right),\ i = 1,2,\ t \le T.$$

The optimal strategy π^* for a given c.c. H is defined

from $El\left(\left(H - X_T^{\pi^*}\right)^+\right) = \inf_{\pi} El\left(\left(H - X_T^{\pi}\right)^+\right),$

Where the inf is taken over all self-financing strategies with nonnegative values satisfying the budget restriction $X_0^{\pi} \le X_0 < E^* H$.

For the mixed contract with $H(T(x)) = H \cdot I_{\{T(x) > T\}}$ the bound $X_0 = {}_T p_x E^* H = {}_T p_x E^* \max\{S_T^1, S_T^2\}$.

The efficient hedge π^* for this problem exists and coincides with a perfect hedge for a modified c.c. H_p with the structure

$$H_p = H - a_p Z_T^{1/(p-1)} \wedge H \qquad \text{for} \qquad p > 1,$$
$$H_p = H \cdot I_{\{Z_T^{-1} > a_p H^{1-p}\}} \qquad \text{for} \qquad 0 < p < 1,$$
$$H_p = H \cdot I_{\{Z_T^{-1} > a_p\}} \qquad \text{for} \qquad p = 1,$$

where Z is the density of the unique martingale measure P^*, constant a_p is determined from

$$E^* H_p = X_0.$$

We reduce the problem to efficient hedging of the option $\left(S_T^1 - S_T^2\right)^+$ and find the following key relation

$$ {}_T p_x = \frac{E^*\left(S_T^1 - S_T^2\right)_p^+}{E^*\left(S_T^1 - S_T^2\right)^+}, \; p > 0. $$

We give the analysis of this equality. For example, in the case where $0 < p < 1$ and $\frac{\mu_1}{\sigma_1^2} \le 1 - p$, such considerations lead us to the formula:

$$ {}_T p_x = 1 - \frac{S_0^1 \Phi\left(b_+\left(S_0^1, C\tilde{S}_0^2, T\right)\right) - \tilde{S}_0^2 \Phi\left(b_-\left(S_0^1, C\tilde{S}_0^2, T\right)\right)}{S_0^1 \Phi\left(b_+\left(S_0^1, \tilde{S}_0^2, T\right)\right) - \tilde{S}_0^2 \Phi\left(b_-\left(S_0^1, \tilde{S}_0^2, T\right)\right)} $$

where constant C is the unique solution of the characteristic equation

$$ y^{-\alpha_p} = const \cdot \left((y-1)^+\right)^{1-p}, \; y \ge 0 $$

with $$ -\alpha_p = \frac{\mu_1}{\sigma_1^2} + \frac{\sigma_2}{(\sigma_1 - \sigma_2)}\left(\frac{\mu_1}{\sigma_1^2} - (1-p)\right). $$

6.2. Jump-Diffusion Model.

$$dS_t^i = S_{t-}^i\left(\mu_i dt + \sigma_i dW_t - \nu_i d\Pi_t\right),$$
$$i = 1, 2,\ t \le T,\ \sigma_i > 0,\ \nu_i < 1.$$

Here, W is a Wiener process and Π is a Poisson process with intensity $\lambda > 0$.

In the framework of this model, we can find the unique martingale measure P^* with the density

$$Z_T = \exp\left\{\alpha^* W_T - \frac{\left(\alpha^*\right)^2}{2} T + \left(\lambda - \lambda^*\right) T + \Pi_T \left(\ln \frac{\lambda^*}{\lambda} \right)\right\}$$

where

$$\alpha^* = \frac{\mu_2 \nu_1 - \mu_1 \nu_2}{\sigma_2 \nu_1 - \sigma_1 \nu_2},\quad \lambda^* = \frac{\mu_1 \sigma_2 - \mu_2 \sigma_1}{\sigma_2 \nu_1 - \sigma_1 \nu_2}.$$

We can consider the same pricing problems for the given model. For example, quantile methodology leads to the key relation

$$ {}_T p_x = \frac{E^*\left(S_T^1 - S_T^2\right)^+ \cdot I_{A^*}}{E^*\left(S_T^1 - S_T^2\right)^+} $$

where A^* is the maximal set of successful hedging for $\left(S_T^1 - S_T^2\right)^+$.

In case $-2\alpha^*/\sigma_1 \le 1$, we have the following expression for ${}_T p_x$ in terms of *Margrabe's* formula averaged by Poisson distribution:

$$ 1 - \frac{\sum_0^\infty p_{n,T}^* \left[\tilde{S}_{0,n}^1 \Phi\left(b_+\left(\tilde{S}_{0,n}^1, C\tilde{S}_{0,n}^2, T\right)\right) - \tilde{S}_{0,n}^2 \Phi\left(b_-\left(\tilde{S}_{0,n}^1, C\tilde{S}_{0,n}^2, T\right)\right)\right]}{\sum_0^\infty p_{n,T}^* \left[\tilde{S}_{0,n}^1 \Phi\left(b_+\left(\tilde{S}_{0,n}^1, \tilde{S}_{0,n}^2, T\right)\right) - \tilde{S}_{0,n}^2 \Phi\left(b_-\left(\tilde{S}_{0,n}^1, \tilde{S}_{0,n}^2, T\right)\right)\right]} $$

where

$$ p_{n,T}^* = e^{-\lambda^* T} \frac{\left(\lambda^* T\right)^n}{n!}, $$

$$ \tilde{S}_{0,n}^i = S_0^i (1 - \nu_i)^n \exp\left(\nu_i \lambda^* T\right), $$

and C is the unique solution of the equation

$$ y^{-\frac{2\alpha^*}{\sigma_1}} = const \cdot (y-1)^+, \quad y \ge 0. $$

6.3. Quantile hedging in two-factors model generated by correlated Wiener processes.

$$dS_t^i = S_t^i\left(\mu_i dt + \sigma_i dW_t^i\right),$$

$$i = 1, 2,\ t \le T,\ \sigma_1 > \sigma_2 > 0,\ \operatorname{cov}\left(W_t^1, W_t^2\right) = \rho t.$$

This market is complete and the unique martingale measure P^* has a density

$$Z_T = \exp\left\{\sum_{i=1}^{2}\varphi_i W_T^i - \frac{1}{2}\left(\varphi_1^2 + \varphi_2^2 + 2\rho\varphi_1\varphi_2\right)T\right\},$$

$$\varphi_1 = \frac{\rho\mu_2\sigma_1 - \mu_1\sigma_2}{\sigma_1\sigma_2\left(1-\rho^2\right)},\quad \varphi_2 = \frac{\rho\mu_1\sigma_2 - \mu_2\sigma_1}{\sigma_1\sigma_2\left(1-\rho^2\right)}.$$

Both assets S^1 and S^2 are martingales w.r. to P^*. Exploiting this measure, we reproduce the same quantile technique for pricing

$$H(T(x)) = \max\left\{S_T^1, S_T^2\right\}\cdot I_{\{T(x) > T\}}$$

as in Section 4.

Firstly, we find the initial price of this contract as

$$ {}_T U_x = {}_T p_x S_0^2 + {}_T p_x E^* \left(S_T^1 - S_T^2 \right)^+ $$

reducing the problem to quantile hedging of $\left(S_T^1 - S_T^2 \right)^+$. *Secondly*, the quantile hedge π^* is a perfect hedge for the modified c.c. $\left(S_T^1 - S_T^2 \right)^+ I_{A^*}$ where A^* is the maximal set of successful hedging for $\left(S_T^1 - S_T^2 \right)^+$.

These considerations again lead to the key relation

$$ {}_T p_x = \frac{E^* \left(S_T^1 - S_T^2 \right)^+ \cdot I_{A^*}}{E^* \left(S_T^1 - S_T^2 \right)^+}. $$

The set A^* can be represented as

$$ A^* = \left\{ Y_T^{-\alpha} \geq const \cdot \left(Y_T - 1 \right)^+ \right\} $$

with $\alpha = \dfrac{\varphi_1}{\sigma_1} = -1 - \dfrac{\varphi_2}{\sigma_2}$.

We again introduce the characteristic equation

$$y^{-\alpha} = const \cdot (y - 1)^{+}$$

and find that for $-\alpha \le 1$ this equation has the unique solution $C \ge 1$, and for $-\alpha > 1$ there are two solutions $1 \le C_1 < C_2$.

We consider only the first case and find that

$$A^{*} = \{Y_T \le C\}.$$

Using log-normality of Y_T and *Margrabe's* formula for $E^{*}\left(S_T^1 - S_T^2\right)^{+}$, we obtain

$${}_T p_x = 1 - \frac{S_0^1 \Phi\left(b_+\left(S_0^1, CS_0^2, T\right)\right) - S_0^2 \Phi\left(b_-\left(S_0^1, CS_0^2, T\right)\right)}{S_0^1 \Phi\left(b_+\left(S_0^1, S_0^2, T\right)\right) - S_0^2 \Phi\left(b_-\left(S_0^1, S_0^2, T\right)\right)}$$

where

$$b_{\pm}\left(S_0^1, CS_0^2, T\right) = \frac{\ln \frac{S_0^1}{CS_0^2} \pm \frac{\sigma^2 T}{2}}{\sigma \sqrt{T}},$$

$$\sigma^2 = \sigma_1^2 + \sigma_2^2 - 2\sigma_1 \sigma_2 \rho.$$

Applying this to the pricing of premium ${}_{T}U_{x}$, we arrive to the following equation

$$ {}_{T}U_{x} = \left[1 - \frac{S_0^1\Phi\left(b_+\left(S_0^1, CS_0^2, T\right)\right) - S_0^2\Phi\left(b_-\left(S_0^1, CS_0^2, T\right)\right)}{S_0^1\Phi\left(b_+\left(S_0^1, S_0^2, T\right)\right) - S_0^2\Phi\left(b_-\left(S_0^1, S_0^2, T\right)\right)}\right] $$

$$ \times\left[S_0^1\Phi\left(b_+\left(S_0^1, S_0^2, T\right)\right) - S_0^2\Phi\left(b_-\left(S_0^1, S_0^2, T\right)\right) + S_0^2\right]. $$

Hedging strategy π^* with the capital

$$ dX_t^{\pi^*} = \gamma_1^* dS_t^1 + \gamma_2^* dS_t^2 $$

can be calculated in a similar way. Using the independence of increments of Wiener processes, we calculate $X_t^{\pi^*}$ as the conditional expected value

$$ E^*\left(\left(S_T^1 - S_T^2\right)^+ \cdot I_{A^*} \middle| F_t\right), \ t \le T, $$

and get

$$ \gamma_1^* = \Phi\left(b_+\left(S_t^1, S_t^2, T-t\right)\right) - \Phi\left(b_+\left(S_t^1, CS_t^2, T-t\right)\right), $$
$$ \gamma_2^* = -\left[\Phi\left(b_-\left(S_t^1, S_t^2, T-t\right)\right) - \Phi\left(b_-\left(S_t^1, CS_t^2, T-t\right)\right)\right]. $$

7. Concluding Remarks.

1. Some similar calculations can be done for Black-Scholes model with stochastic volatility as a representative model of incomplete market.

2. Further developments of this issue may include the effect of transaction costs.

3. We considered conditioned contingent claims under assumption that "market" and "conditioned factor" τ are independent. This is not necessary. One can study conditioned contingent claims thinking about τ as a source of insider information:

$$F_t \mapsto F_t^{\tau}.$$

As a measure of optimality of the strategy, the criteria of expected utility can be chosen.

4. There exists a close connection with defaultable derivatives and credit risks.

8. References

AASE, K. and PERSSON, S. 1994. "Pricing of unit-linked insurance policies." Scandinavian Actuarial Journal 1: 26-52.

BACINELLO, A.R. and ORTU, F. 1993. "Pricing of unit-linked life insurance with endegeneous minimum guarantees." Insurance: Math. and Economics 12:245-257.

BOWERS, N.L., GERBER, H.U., HICKMAN, J.C., JONES D.A. and NESBITT, C.I. 1997. "Actuarial Mathematics." Society of Actuaries, Schaumburg, Illinois.

BOYLE, P.P. and SCHWARTZ, E.S. 1977. "Equilibrium prices of guarantees under equity-linked contracts." Journal of Risk and Insurance 44: 639-680.

BOYLE, P.P. and HARDY, M.R. 1997. "Reserving for maturity guarantees: Two approaches." Insurance: Math. and Economics 21: 113-127.

BRENNAN, M.J. and SCHWARTZ, E.S. 1976. "The pricing of equity-linked life insurance policies with an asset value guarantee." Journal of Financial Economics 3: 195-213.

BRENNAN, M.J. and SCHWARTZ, E.S. 1979. "Alternative investment strategies for the issuers of equity-linked life insurance with an asset value guarantee." Journal of Business 52: 63-93.

DELBAEN, F. 1986. "Equity-linked policies." Bulletin Association Royal Actuaries Belges 80: 33-52.

FOELLMER, H. and LEUKERT, P. 2000. “Efficient hedging: cost versus short-fall risk.” Finance Stochast. 4: 117-146.

FOELLMER, H. and SCHIED, A. 2002. “Stochastic Finance: An introduction in discrete time.” Berlin – N.Y.: Walter de Gruyter.

HARDY, M.R. 2003. “Investment guarantees: Modeling and risk-management for equity-linked life insurance.” J.Wiley.

MARGRABE, W. 1978. “The value of an option to exchange one asset to another.” J. of finance 33: 177-186.

MELNIKOV, A., VOLKOV, S. and NECHAEV, M. 2002. “Mathematics of Financial Obligations.” American Math. Soc.

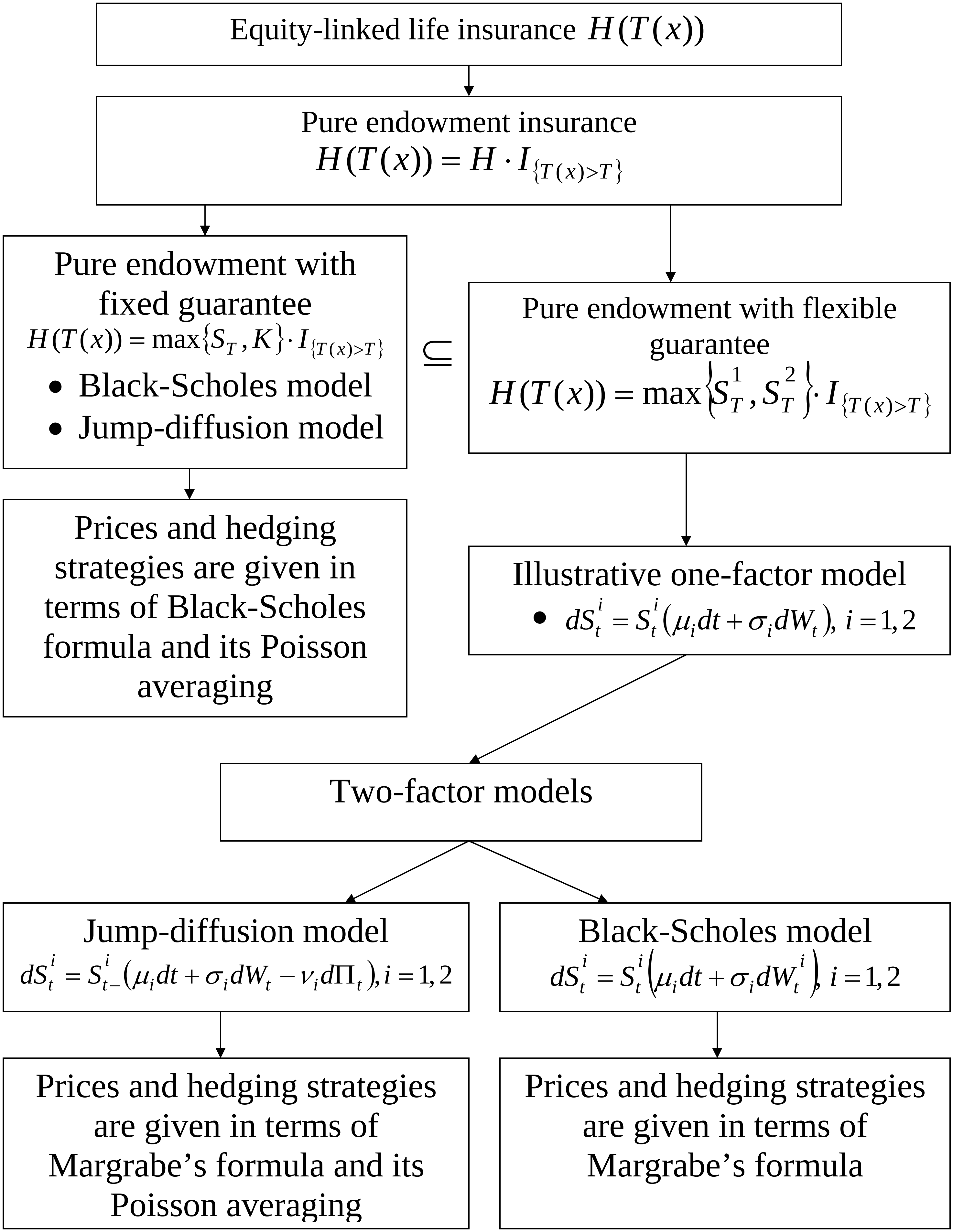
MELNIKOV, A. 2003. “Risk analysis in Finance and Insurance.” Chapman&Hall/CRC.

MELNIKOV, A. 2004.”Quantile hedging of equity-linked life insurance policies” Doklady Mathematics (Proceedings of Russian Acad.Sci).

MELNIKOV, A. 2004.”On efficient hedging of equity-linked life insurance policies” Doklady Mathematics (Proceedings of Russian Acad.Sci.).

MOELLER, T. 1998. “Risk-minimizing hedging strategies for unit-linked life-insurance contracts.” Astin Bulletin 28: 17-47.

MOELLER, T. 2002. “Hedging equity-linked life insurance contracts.” North American Actuarial Journal 5: 79-95.


Equity-linked life insurance $H(T(x))$
Pure endowment insurance
$H(T(x)) = H \cdot I_{\{T(x)>T\}}$
Pure endowment with fixed guarantee
$H(T(x)) = \max\{S_T, K\} \cdot I_{\{T(x)>T\}}$
• Black-Scholes model
• Jump-diffusion model
$\subseteq$
Pure endowment with flexible guarantee
$H(T(x)) = \max\{S_T^1, S_T^2\} \cdot I_{\{T(x)>T\}}$
Prices and hedging strategies are given in terms of Black-Scholes formula and its Poisson averaging
Illustrative one-factor model
• $dS_t^i = S_t^i(\mu_i dt + \sigma_i dW_t), \; i = 1, 2$
Two-factor models
Jump-diffusion model
$dS_t^i = S_{t-}^i(\mu_i dt + \sigma_i dW_t - \nu_i d\Pi_t), i = 1, 2$
Black-Scholes model
$dS_t^i = S_t^i(\mu_i dt + \sigma_i dW_t^i), \; i = 1, 2$
Prices and hedging strategies are given in terms of Margrabe's formula and its Poisson averaging
Prices and hedging strategies are given in terms of Margrabe's formula