Basic Economic Scenario Generator: Technical Specifications

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Contents

Introduction

This document presents all models and assets included in the basic economic scenario generator. Each chapter contains closed formulas and coding specifications associated (discretization methods and containing file).

Chapter 1

Risk factors models

In this first chapter, different risk factors and their respective models are presented. Each model is theoretically detailed with specific coding methods when it's needed.
In this economic scenario generator (ESG), all risk factors are modeled under the risk-neutral measure. The short rate is stochastic and its diffusion process follows the Vasicek Extended model (also known as Hull-White model). Stocks and real-estate assets are driven through Black-Scholes with stochastic short rate model. Default and liquidity spreads are modeled under the Longstaff Mithal and Neis (LMN) model which consider default as Poisson process increment.
No correlation are considered except a linear relation between stocks and zero-coupon bonds.

1.1 Conventions

In order to simplify this document's reading, a few conventions are assumed:

- $f(t,T)$ is the forward rate beginning in t and finishing in T,
- r_t is the short rate in t,
- $P(t,T)$ is the zero-coupon bond price in t giving a unit in T,
- ϵ is a standard gaussian random variable.

1.2 Short rate

Under Solvency 2, economic scenario generators have to take the interest rate term structure as an input for short rate calibration. This constraint is respected in the Vasicek extended model, which is a direct application of Heath, Jarrow and Morton (HJM) framework (see the annex of this document for more information). This simple model is convenient as it provides closed formula for bonds (see assets valuation chapter).
In order to clarify formulas, these additional conventions are added:

$$\begin{aligned} K(t) &= \frac{1-e^{-kt}}{k}, \\ L(t) &= \frac{\sigma^2}{2k}(1-e^{-2kt}). \end{aligned} \tag{1.1}$$

1.2.1 Vasicek extended model

Vasicek extended model is derived from the HJM approach (cf. technical annex) with a diffusion process for the forward rate as follows:

$$df(t,T) = \mu_f(t,T)dt + \sigma_f(t,T)dW_t^1 \tag{1.2}$$

with W_t^1 a standard brownian motion under the risk-neutral measure. In this particular model, the volatility structure is specified in order to exponentially decrease with maturity:

$$\sigma_f(t,T) = \sigma e^{-k(T-t)} \tag{1.3}$$

From this expression, one can derive the forward rate's trend (cf technical annex):

$$\mu_f(t,T) = \sigma_f(t,T) \int_t^T \sigma_f(t,u)du = \frac{\sigma^2}{k}\left(e^{-k(T-t)} - e^{-2k(T-t)}\right) \tag{1.4}$$

Now that the trend and the volatility structure are known, integration of equation 1.2 between t and T provides:

$$\begin{aligned} f(t,T) &= f(0,T) + \int_0^t \mu_f(s,T)ds + \int_0^t \sigma_f(s,T)d\hat{W}_s^1, \\ f(t,T) &= f(0,T) + \int_0^t \frac{\sigma^2}{k}\left(e^{-k(T-s)} - e^{-2k(T-s)}\right)ds + \int_0^t \sigma e^{-K(T-s)}d\hat{W}_s^1. \end{aligned} \tag{1.5}$$

In which one can immediately derive the short rate:

$$r_t = f(t,t) \tag{1.6}$$

1.2.2 Discretization

Numerical implementation requires discrete equations. A discrete version of equation 1.6 is obtained thereby:

$$r_{t+\delta} - e^{-k\delta}r_t = f(0,t+\delta) - e^{-k\delta}f(0,t) + \frac{\sigma^2}{2}\left(K(t+\delta)^2 - e^{-k\delta}K(t)^2\right) + \sigma e^{-k\delta}\int_t^{t+\delta} e^{-k(t-s)}d\hat{W}_s^1 \tag{1.7}$$

where δ is the discretization step. Finally, the short rate is coded in the file *Methods_PathsGeneration.R* as follows:

$$\boxed{r_{t+\delta} = e^{-k\delta}r_t + f(0,t+\delta) - e^{-k\delta}f(0,t) + \frac{\sigma^2}{2}\left(K(t+\delta)^2 - e^{-k\delta}K(t)^2\right) + \sqrt{L(\delta)}\epsilon} \tag{1.8}$$

1.2.3 Forward rate extraction

As previously mentioned, Vasicek extend model requires the forward rate structure $f(0,t)$ as an input. The approach used in this ESG is based on the equation 1.9:

$$f(0,t) = \lim_{\delta\to 0} \frac{R(0,t+\delta)(t+\delta) - R(0,t)t}{\delta} \tag{1.9}$$

which is computed as follow in *Functions_ForwardExtraction.R*:

$$f(0,t) = \frac{R\left(0,t+\frac{1}{12}\right)\left(t+\frac{1}{12}\right) - R(0,t)t}{\frac{1}{12}} \tag{1.10}$$

The initial term structure is extracted from the file *ZCrates.csv* (obtained from the French Institute of Actuaries, cf User's guide for more information) containing zero-coupon rates at a monthly frequency for the next 40 years.

1.3 Stock and Real-estate

Stock and real-estate assets are modeled under an extended version of Black & Scholes under stochastic short rate [1].

1.3.1 Black & Scholes model under stochastic short rate

In this ESG, stock and real-estate follow the same diffusion process:

$$\begin{aligned}\frac{dS_t}{S_t} &= r_t dt + \rho\sigma_S dW_t^2, \\ \frac{dS_t}{S_t} &= r_t dt + \rho\sigma_S d\hat{W}_t^1 + \sqrt{1-\rho^2}\sigma_S d\hat{W}_t^2\end{aligned} \tag{1.11}$$

where $\hat{W}_t^1$ and $\hat{W}_t^2$ are independent brownian motions obtained by Cholesky decomposition. Itô's lemma drives to the solution:

$$S_t = S_0 \exp\left(\int_0^t r_u du - \frac{\sigma_S^2}{2}t + \rho\sigma_S\hat{W}_t^1 + \sqrt{1-\rho^2}\sigma_S\hat{W}_t^2\right) \tag{1.12}$$

1.3.2 Discretization

In order to numerically generate paths for S, equation 1.12 needs to be discretized and is coded in *Methods_PathsGeneration.R* as follows:

$$\boxed{S_{t+\delta} = S_t \exp\left(\left(r_{t+\delta} - \frac{\sigma_S^2}{2}\right)\delta + \rho\sigma_S\sqrt{\delta}\epsilon_1 + \sqrt{1-\rho^2}\sigma_S\sqrt{\delta}\epsilon_2\right)} \tag{1.13}$$

Note that this discretization doesn't create any bias.

1.4 Credit and liquidity risks

In this ESG, credit and liquidity risks are considered in order to include default-able bonds and credit default swaps (CDS). The LMN model considers credit and liquidity event as the result of a Poisson process with stochastic intensities [4].
The credit intensity λ is driven through the CIR model (Cox, Ingersoll et Ross [2]) wich impose positivity, mean attraction and heteroscedasticity:

$$d\lambda = (\alpha - \beta\lambda)dt + \sigma_\lambda \lambda d\hat{W}_t^4 \tag{1.14}$$

where α,β and σ_λ are positive real numbers. Liquidity risk intensity is modeled as a simple white noise:

$$d\gamma_t = \eta d\hat{W}_t^3 \tag{1.15}$$

where η is a positive real number. Each notation grade (AAA, AA, etc.) is then modeled by different coefficient values.

1.4.1 Discretization

The two intensity processes need to be discretized in order to be numerically generated in the ESG and are coded in *Methods_PathsGeneration.R*. Discretization is direct and exact for liquidity:

$$\boxed{\gamma_{t+\delta} = \gamma_t + \eta\sqrt{\delta}\epsilon} \tag{1.16}$$

Whereas default intensity requires the use of the Milstein scheme (which is a second order Itô-taylor development):

$$\boxed{\lambda_{t+\delta} = \lambda_t + \alpha(\beta - \lambda_t)\delta + \sigma_\lambda\sqrt{\lambda_t\delta}\epsilon + \frac{\sigma_\lambda^2}{4}\delta(\epsilon^2 - 1)} \tag{1.17}$$

Chapter 2

Assets valuation

In this chapter, multiple products are introduced and priced according to models presented in chapter 1. The discounting factor is define thereby:

$$\delta(T) = \exp\left(-\int_0^T r_u du\right) \tag{2.1}$$

2.1 Basic assets

In this section, default-free zero-coupon bonds and default-free ordinary bonds are presented.

2.1.1 Default-free zero-coupon bond

Zero-coupon prices are directly derived from the expression of the forward rate as follows:

$$P(t,T) = E\left(\frac{\delta(T)}{\delta(t)}\right) = \exp\left(-\int_t^T f(t,u)du\right) \tag{2.2}$$

As the forward rate dynamic is governed by Vasicek extended model, the zero-coupon price is computed in *Methods_PriceDistribution_ZCBond* thereby:

$$\boxed{P(t,T) = \frac{P(0,T)}{P(0,t)} \exp\left(-\frac{K(T-t)^2}{2}L(t) + K(T-t)(f(0,t) - r_t)\right)} \tag{2.3}$$

This last formula requires the prices $P(0,t)$ and $P(0,T)$. These two values are obtained directly from the zero-coupon rate $R(0,t)$ in *ZCrates.csv* as follows:

$$P(0,t) = \exp\left(-R(0,t)t\right) \tag{2.4}$$

In the file *ZCrates.csv*, zero-coupon rates are given monthly. In order to extend the possible maturities in the ESG, a linear extrapolation is realized for zero-coupon bond prices:

$$P(0,t+\delta) = P(0,t) + \delta\frac{P(0,t+\frac{1}{12}) - P(0,t)}{\frac{1}{12}} \tag{2.5}$$

with $\delta \in [0, \frac{1}{12}]$.

2.1.2 Default-free ordinary bond

Default-free bonds can be considered as a sum of zero-coupon bonds with different maturities (yearly coupons in this example):

$$P(t,T) = P(t,T) + c\sum_{i=t+1}^{T} P(t,i) \tag{2.6}$$

where c is the coupon rate. In the ESG, maturities between coupon dates are modeled and it's necessary to add coupon's time value to the clean price to get the bond dirty price:

$$\text{Dirty price} = \text{Clean price} + \text{Coupon's time value} \tag{2.7}$$

The coupon's time value is the next coupon at a prorata temporis value. Considering c the bond coupon rate, n_c the number of coupons each year, T the maturity and t the valuation date, the clean price is computed as follows:

$$\begin{aligned} C_p(c, n_c, t, T) &= P(t,T) + c \sum_{i=0}^{\beta(t)-1} P(t, \alpha_i) \\ \beta(t) &= \lfloor n_c(T-t) + 1 \rfloor \\ \alpha_i &= T - \frac{i}{n_c} \end{aligned} \tag{2.8}$$

where α_i is the $(\beta(t)-i)$-th coupon's date and $\beta(t)$ the number of remaining coupons. In this ESG, maturity is considered to be equal to the last coupon date to avoid additional technical difficulties. Coupon's time value is computed thereby:

$$C_c(c, n_c, t, T) = c\left(t - \alpha_{\beta(t)} + \frac{3}{360}\right) \tag{2.9}$$

where $\frac{3}{360}$ represents the market 3 days of delay. Finally, the dirty price is coded as follows in *Methods_PriceDistribution_Bond.R*:

$$\boxed{O(c, n_c, t, T) = C_p(c, n_c, t, T) + C_c(c, n_c, t, T)} \tag{2.10}$$

2.2 Default-able assets

In this section, all default-able assets are valuated under chapter 1 diffusions.

2.2.1 Corporate bond

In the LMN model [4], corporate bonds are valuated thereby:

$$\begin{aligned} CB(c, \omega, T) = c &\int_0^T A(u)\exp(B(u)\lambda_0)C(u)P(0,u)\exp(-\gamma_0 u)du \\ &+ A(T)\exp(B(T)\lambda_0)C(T)P(0,T)\exp(-\gamma_0 T) \\ &+ (1-\omega)\int_0^T \exp(B(u)\lambda_0)C(u)P(0,u)(G(u)+\lambda_0 H(u))\exp(-\gamma_0 u)du \end{aligned} \tag{2.11}$$

where functions A,B,C,G and H are detailed in annex and ω is the part still paid in case of default. Considering discrete coupons and changing the date of valuation, clean price formula is derived from equation 2.11 thereby:

$$\begin{aligned} CB_p(c, n_c, t, T, \omega) = c &\sum_{i=0}^{\beta(t)-1} A(\alpha_i)\exp(B(\alpha_i)\lambda_t)C(\alpha_i)P(t,\alpha_i)\exp(-\gamma_t \alpha_i) \\ &+ A(T)\exp(B(T)\lambda_t)C(T)P(t,T)\exp(-\gamma_t T) \\ &+ (1-\omega)\int_0^{T-t} \exp(B(u)\lambda_t)C(u)P(t,u)(G(u)+\lambda_t H(u))\exp(-\gamma_t u)du \end{aligned} \tag{2.12}$$

Same notations and behavior are used for the coupon's time value (nor default nor liquidity spread considered). Finally, the market value for a corporate bond is given by the following formula:

$$
\begin{aligned}
CB_p(c, n_c, t, T, \omega) = c &\sum_{i=0}^{\beta(t)-1} A(\alpha_i)\exp(B(\alpha_i)\lambda_t)C(\alpha_i)P(t,\alpha_i)\exp(-\gamma_t\alpha_i) \\
&+ A(T)\exp(B(T)\lambda_t)C(T)P(t,T)\exp(-\gamma_t T) \\
&+ (1-\omega)\int_0^{T-t}\exp(B(u)\lambda_t)C(u)P(t,u)(G(u)+\lambda_t H(u))\exp(-\gamma_t u)du + C_c(\rho, n_c, t, T)
\end{aligned} \tag{2.13}
$$

Discretization

In the ESG, corporate bonds are coded in the file *Methods_PriceDistribution_CorporateBond.R.* Discretization is required for integral calculus with n steps and $\delta = \frac{T-t}{n}$. Considering that the coupons are discrete and not continuous, the valuation formula becomes:

$$
\begin{aligned}
CB(c, n_c, t, T, \omega) = c &\sum_{i=0}^{\beta(t)-1} A(\alpha_i)\exp(B(\alpha_i)\lambda_t)C(\alpha_i)P(t,\alpha_i)\exp(-\gamma_t\alpha_i) \\
&+ A(T)\exp(B(T)\lambda_t)C(T)P(t,T)\exp(-\gamma_t T) \\
&+ (1-\omega)\sum_{i=1}^{n}\delta\exp(B(i\delta)\lambda_t)C(i\delta)P(t,t+i\delta)(G(i\delta)+\lambda_t H(i\delta))\exp(-\gamma_t i\delta) \\
&+ C_c(c, n_c, t, T)
\end{aligned} \tag{2.14}
$$

2.2.2 Credit Default Swap

Credit default swap is valuated through the LMN model [4]. The premium s is supposed continuous and is the fair value between the insurer and the protection buyer. The valuation formula at initial date 0 is presented thereby:

$$
s(\omega, 0, T) = \frac{\omega\int_0^T \exp(B(t)\lambda_0)P(0,t)(G(t)+H(t)\lambda_0)dt}{\int_0^T A(t)\exp(B(t)\lambda_0)P(0,t)dt} \tag{2.15}
$$

In the ESG, CDS don't necessarily start at date 0, thus the previous formula is adapted as follows:

$$
s(\omega, t, T) = \frac{\omega\int_0^{T-t} \exp(B(u)\lambda_t)P(t,u)(G(u)+H(u)\lambda_t)du}{\int_0^{T-t} A(u)\exp(B(u)\lambda_t)P(t,u)du} \tag{2.16}
$$

Discretization

The CDS continuous premium is computed in the file *Methods_PriceDistribution_CDSPremium.R.* Formula 2.16 requires discretization for the integrals calculus with n steps $\delta = \frac{T-t}{n}$:

$$
s(\omega, t, T) = \frac{\omega\sum_{i=1}^{n}\delta\exp(B(i\delta)\lambda_t)P(t,t+i\delta)(G(i\delta)+H(i\delta)\lambda_t)}{\sum_{i=1}^{n}\delta A(i\delta)\exp(B(i\delta)\lambda_t)P(t,t+i\delta)} \tag{2.17}
$$

2.3 Derivatives

In this section, vanilla derivatives products are valuated.

2.3.1 Short-rate European derivative

In this section, European calls and puts on zero-coupon bonds are valuated. The price of a call on a zero-coupon bond starting at T with maturity s at date t is valuated thereby (see annex for details):

$$\begin{aligned} C(t,T,s,K) &= P(t,s)N(d_1) - KP(t,T)N(d_2), \\ d_1 &= \frac{1}{H(T,s)} ln\left(\frac{P(t,s)}{P(t,T)K} + \frac{H(T,s)}{2}\right), \\ d_2 &= d_1 - H(T,s), \\ H(T,s) &= \sqrt{\frac{\sigma^2}{2k^3}\left((e^{-k(s-T)}-1)^2 - (e^{-ks}-e^{-kT})^2\right)} \end{aligned} \tag{2.18}$$

with $t < T \leq s$. This product is coded in the file *Methods_PriceDistribution_EuroCallPut_ZC.R*. Put prices are derived through the Call-Put relation (see annex for details).

2.3.2 Stock European derivative

In this section, European calls and puts on stock are valuated. The call price at date t with maturity T and strike K is computed thereby:

$$\begin{aligned} C(t,T,K) &= S_tN(d_1) - KP(t,T)N(d_2), \\ d_1 &= \frac{ln\left(\frac{S_t}{P(t,T)K}\right) + \frac{1}{2}\tau(T-t)}{\sqrt{\tau(T-t)}}, \\ d_2 &= d_1 - \sqrt{\tau(T-t)}, \\ \tau(T) &= \left(\frac{\sigma^2}{k^2} + \frac{2\rho\sigma\sigma_S}{k} + \sigma_S^2\right)T + 2\left(\frac{\sigma^2}{k^2} - \frac{\rho\sigma\sigma_S}{k}\right)K(T) - \frac{L(T)}{K^2}. \end{aligned} \tag{2.19}$$

This formula is detailed in annex and can be found in [3].

2.3.3 Convertible bond

A convertible bond is composed of a bond (non-default-able in this ESG) and a call on stock with same maturity and strike equal to the bond principal. The valuation is computed from this equation:

$$OC(c,t,T) = c(O(\rho,t,T) + C(t,T,1)) \tag{2.20}$$

where α is an adjustment parameter.

Bibliography

[1] Scholes M. Black F. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81(3):637–654, 1973.

[2] Ross S. Cox J., Ingersoll J. A theory of the term structure of interest rates. *Econometrica*, 53:385–408, 1985.

[3] Quittard-Pinon F. *Mathématiques financières*. EMS, 2002.

[4] Neis E. Longstaff F., Mithal S. Corporate yield spreads: Default risk or liquidity? new evidence from the credit-default swap market. *NBER Working paper*, (10418), April 2004.

Appendix A

Technical annex

This technical annex presents detailed demonstration for risk factor models and assets valuation.

A.1 Support functions

This section details functions used for CDS and corporate bonds valuation:

$$\begin{aligned}
\phi &= \sqrt{2\sigma^2 + \beta^2} \\
\kappa &= \frac{\beta + \phi}{\beta - \phi} \\
A(t) &= \exp\left(\frac{\alpha(\beta + \phi)}{\sigma^2} t\right)\left(\frac{1-\kappa}{1-\kappa e^{\phi t}}\right)^{\frac{2\alpha}{\sigma^2}} \\
B(t) &= \frac{\beta - \phi}{\sigma^2} + \frac{2\phi}{\sigma^2(1-\kappa e^{\phi t})} \\
C(t) &= \exp\left(\frac{\eta^2 t^3}{6}\right) \\
G(t) &= \frac{\alpha}{\phi}\left(e^{\phi t} - 1\right)\exp\left(\frac{\alpha(\beta + \phi)}{\sigma^2} t\right)\left(\frac{1-\kappa}{1-\kappa e^{\phi t}}\right)^{\frac{2\alpha}{\sigma^2}+1} \\
H(t) &= \exp\left(\frac{\alpha(\beta + \phi) + \phi\sigma^2}{\sigma^2} t\right)\left(\frac{1-\kappa}{1-\kappa e^{\phi t}}\right)^{\frac{2\alpha}{\sigma^2}+2}
\end{aligned} \tag{A.1}$$

A.2 Heath, Jarrow and Morton (HJM) approach

In their article, Heath, Jarrow and Morton establish a method to model interest rates term structure through the no-free lunch hypothesis. To do so, they suppose the following diffusion process for the forward rate:

$$df(t,T) = \mu_f(t,T)dt + \sigma_f(t,T)d\hat{W}_t^1 \tag{A.2}$$

with W_t a standard brownian motion under the risk-neutral probability. In this framework, authors suppose a Black & Scholes type diffusion for the zero-coupon bond:

$$\frac{dP(t,T)}{P(t,T)} = r_t dt - \sigma_P(t,T)d\hat{W}_t^1 \tag{A.3}$$

This relation leads to a link between the trend and the volatility structure of the forward rate. Indeed, a zero-coupon bond is valuated as follows:

$$
\begin{aligned}
P(t,T) &= \exp\left(-\int_t^T f(t,u)du\right), \\
\ln(P(t,T)) &= -\int_t^T f(t,u)du.
\end{aligned} \tag{A.4}
$$

Then, through the Leibnitz rule:

$$
\begin{aligned}
d\ln(P(t,T)) &= f(t,t)dt - \int_t^T df(t,u)du, \\
&= r_t dt - \int_t^T \left(\mu_f(t,u)dt + \sigma_f(t,u)d\hat{W}_t^1\right) du, \\
&= \left(r_t - \int_t^T \mu_f(t,u)du\right) dt - \left(\int_t^T \sigma_f(t,u)du\right) d\hat{W}_t^1.
\end{aligned} \tag{A.5}
$$

The Itô's lemma gives us:

$$
\begin{aligned}
d\ln(P(t,T)) &= \frac{1}{P(t,T)}dP(t,T) - \frac{1}{2P(t,T)^2}d < P(t,T) >, \\
&= \left(r_t - \frac{1}{2}\sigma_P(t,T)^2\right) dt - \sigma_P(t,T)d\hat{W}_t^1
\end{aligned} \tag{A.6}
$$

which leads to the following equalities by identification:

$$
\begin{aligned}
&\int_t^T \mu_f(t,u)du = \frac{1}{2}\sigma_P(t,T)^2, \\
&\int_t^T \sigma_f(t,u)du = \sigma_P(t,T)
\end{aligned} \tag{A.7}
$$

Thus:

$$
\int_t^T \mu_f(t,u)du = \frac{1}{2}\left(\int_t^T \sigma_f(t,u)du\right)^2 \tag{A.8}
$$

Finally,

$$
\boxed{\mu_f(t,T) = \sigma_f(t,T)\int_t^T \sigma_f(t,u)du} \tag{A.9}
$$

A.3 European call on zero-coupon bonds

In this section, the valuation of European zero-coupon bonds is detailed into the Vasicek extended model framework.

A.3.1 No free-lunch valuation

Under this principle, with $s > T$, the call price becomes:

$$
C(0,T,K) = E_Q((P(T,s) - K)^+\delta(T)) \tag{A.10}
$$

With the T-forward probability associated to the following Radon-Nykodin density:

$$\frac{dQ_T}{dQ} = \frac{\delta(T)}{P(0,T)} \tag{A.11}$$

It comes with $a = \{P(T,s) \geq K\}$:

$$\begin{aligned} C(0,T,K) &= E_Q((P(T,s) - K)^+ \delta(T)) \\ &= P(0,T) E_{Q_T}\left((P(T,s) - K)^+\right) \\ &= P(0,T)(E_{Q_T}(P(T,s)\mathbb{1}_a) - KQ_T(a)) \end{aligned} \tag{A.12}$$

A.3.2 $P(T,s)$ law under T-forward measure

Using Itô's lemma in A.5, it comes:

$$P(T,s) = P(0,s) \exp\left(\int_0^T r_u du - \int_0^T \sigma_P(u,s) d\hat{W}_u^1 - \frac{1}{2}\int_0^T \sigma_P^2(u,s) du\right) \tag{A.13}$$

To bring back this equation under the T-forward measure, we use the following Radon-Nykodin measure:

$$\begin{aligned} \frac{dQ_T}{dQ} &= \frac{P(T,T)}{P(0,T)}\delta(T) \\ &= \exp\left(\int_0^T r_u du - \int_0^T \sigma_P(u,s) d\hat{W}_u^1 - \frac{1}{2}\int_0^T \sigma_P^2(u,s) du\right)\delta(T) \\ &= \exp\left(-\int_0^T \sigma_P(u,s) d\hat{W}_u^1 - \frac{1}{2}\int_0^T \sigma_P^2(u,s) du\right) \end{aligned} \tag{A.14}$$

Then, Girsanov theorem identifies a T-brownian motion $\tilde{W}_t$:

$$d\tilde{W}_u = d\hat{W}_u + \sigma_P(u,s) du \tag{A.15}$$

Observing these two relations:

$$\begin{aligned} P(T,s) &= P(0,s) \exp\left(\int_0^T r_u du - \int_0^T \sigma_P(u,s) d\hat{W}_u^1 - \frac{1}{2}\int_0^T \sigma_P^2(u,s) du\right) \\ P(T,T) &= P(0,T) \exp\left(\int_0^T r_u du - \int_0^T \sigma_P(u,T) d\hat{W}_u^1 - \frac{1}{2}\int_0^T \sigma_P^2(u,T) du\right) \end{aligned} \tag{A.16}$$

As $P(T,T) = 1$, the first line can be rewritten using the second one:

$$\begin{aligned} P(T,s) &= \frac{P(0,s)}{P(0,T)} \exp\left(\int_0^T -[\sigma_P(u,s) - \sigma_P(u,T)] d\hat{W}_u^1 - \frac{1}{2}\int_0^T [\sigma_P^2(u,s) - \sigma_P^2(u,T)] du\right) \\ P(T,s) &= \frac{P(0,s)}{P(0,T)} \exp\left(\int_0^T -[\sigma_P(u,s) - \sigma_P(u,T)](d\tilde{W}_u^1 - \sigma_P(u,s) du) - \frac{1}{2}\int_0^T [\sigma_P^2(u,s) - \sigma_P^2(u,T)] du\right) \\ P(T,s) &= \frac{P(0,s)}{P(0,T)} \exp\left(\int_0^T -[\sigma_P(u,s) - \sigma_P(u,T)] d\tilde{W}_u^1 - \frac{1}{2}\int_0^T [\sigma_P(u,s) - \sigma_P(u,T)]^2 du\right) \end{aligned} \tag{A.17}$$

With:

$$
\begin{aligned}
W_T &= \int_0^T -[\sigma_P(u,s) - \sigma_P(u,T)]dz_t(u) - \frac{1}{2}\int_0^T [\sigma_P(s,T) - \sigma_P(0,T)]^2 du \\
H^2 &= \int_0^T [\sigma_P(u,s) - \sigma_P(u,T)]^2 du
\end{aligned} \tag{A.18}
$$

One can observe that W_T is distributed according to $\mathcal{N}(-\frac{H^2}{2}, H)$ under the T-forward measure. Then:

$$
P(T,s) = \frac{P(0,s)}{P(0,T)}\exp(W_T) \tag{A.19}
$$

a can be rewritten as $\{\exp(W_T) \geq K\frac{P(0,T)}{P(0,s)}\}$, so:

$$
\boxed{
\begin{aligned}
C(0,T,K) &= P(0,s)N(d_1) - KP(0,T)N(d_2) \\
d_1 &= \frac{1}{H}ln\left(\frac{P(0,s)}{P(0,T)K} + \frac{H}{2}\right) \\
d_2 &= d_1 - H
\end{aligned}} \tag{A.20}
$$

A.3.3 European put on zero-coupon bonds

Call-Put parity applies to european options:

$$
S + Put = Call + KP(0,T) \tag{A.21}
$$

The put expression derives directly from A.20:

$$
\begin{aligned}
Put(0,T,K) &= Call(0,T,K) - P(0,s) + KP(0,T) \\
&= P(0,s)(N(d_1) - 1) + KP(0,T)(1 - N(d_2)) \\
&= KP(0,T)N(-d_2) - P(0,s)N(-d_1)
\end{aligned} \tag{A.22}
$$

A.3.4 Vasicek extended model application

In this model, the volatility structure is the following:

$$
\sigma_P(t,T) = \int_t^T \sigma e^{-k(u-t)}du = \frac{\sigma}{k}(1 - e^{-k(T-t)}) = \sigma K(T-t) \tag{A.23}
$$

H computation becomes:

$$
\begin{aligned}
H^2 &= \int_0^T [\sigma_P(u,s) - \sigma_P(u,T)]^2 du \\
&= \int_0^T \sigma_P^2(u,s)du - 2\int_0^T \sigma_P(u,s)\sigma_P(u,T)du + \int_0^T \sigma_P^2(u,T)du \\
&= A - 2B + C
\end{aligned} \tag{A.24}
$$

where A,B and C are detailed thereby:

$$
\begin{aligned}
A &= \frac{\sigma^2}{k^2}\left(T - \frac{2}{k}(e^{-k(s-T)} - e^{-ks}) + \frac{1}{2k}(e^{-2k(s-T)} - e^{-2ks})\right) \\
B &= \frac{\sigma^2}{k^2}\left(T - \frac{1}{k}(1 - e^{-kT}) - \frac{1}{k}(e^{-k(s-T)} - e^{-ks}) + \frac{1}{2k}(e^{-k(s-T)} - e^{-k(s+T)})\right) \\
C &= \frac{\sigma^2}{k^2}\left(T - \frac{2}{k}(1 - e^{-kT}) + \frac{1}{2k}(1 - e^{-2kT})\right)
\end{aligned} \tag{A.25}
$$

And finally:

$$\boxed{H = \sqrt{\frac{\sigma^2}{2k^3}\left((e^{-k(s-T)}-1)^2-(e^{-ks}-e^{-kT})^2\right)}} \tag{A.26}$$

A.4 European call on stock

As a reminder, these diffusion are considered:

$$\begin{aligned}\frac{dP}{P} &= r_t dt - \sigma_P(t,T) d\hat{W}_t^1 \\ \frac{dS}{S} &= r_t dt + \rho\sigma_S d\hat{W}_t^1 + \sqrt{1-\rho^2}\sigma_S d\hat{W}_t^2 \\ d\hat{W}_t^1 d\hat{W}_t^2 &= 0\end{aligned} \tag{A.27}$$

A.4.1 No-free lunch principle

The standard valuation method under no-free lunch hypothesis is used:

$$\begin{aligned}C(0,T,K) &= E_Q((S_T-K)^+\delta(T)) \\ &= E_Q(S_T\delta(T)\mathbb{1}_b) - E_Q(K\delta(T)\mathbb{1}_b) \\ &= S_0 Q_S(b) - KP(0,T)Q_T(b)\end{aligned} \tag{A.28}$$

Writing $b = \{S_T \geq K\}$, fist term becomes:

$$\begin{aligned}A &= E_Q(S_T\delta(t,T)\mathbb{1}_b) \\ &= E_Q\left(\frac{S_0}{S_0}S_T\delta(T)\mathbb{1}_b\right) \\ &= S_0 E_{Q_S}(\mathbb{1}_b) \\ &= S_0 Q_S(b)\end{aligned} \tag{A.29}$$

About the second one:

$$\begin{aligned}B &= E_Q(\delta(T)\mathbb{1}_b) \\ &= E_Q(P(0,T)\delta(T)\frac{P(T,T)}{P(0,T)}\mathbb{1}_b) \\ &= P(0,T)Q_T(b)\end{aligned} \tag{A.30}$$

Now, the objective is to evaluate the probability of event b under Q_S and Q_T measures. Itô's lemma gives:

$$S_T = S_0 \exp\left(\int_0^T r_u du\right)\exp\left(\int_0^T (-\frac{\sigma_S^2}{2}du + \rho\sigma_S d\hat{W}_u^1 + \sqrt{1-\rho^2}\sigma_S d\hat{W}_u^2)\right) \tag{A.31}$$

The Radon-Nykodin measure is:

$$\frac{dQ_S}{dQ} = \frac{S_T\delta(T)}{S_0} = \exp\left(\int_0^T (-\frac{\sigma_S^2}{2}du + \rho\sigma_S d\hat{W}_u^1 + \sqrt{1-\rho^2}\sigma_S d\hat{W}_u^2)\right) \tag{A.32}$$

Multi-dimensional Girsanov's theorem shows two brownian motions under Q_S :W_S^1 and W_S^2 as follows:

$$\begin{aligned}\breve{W}_t^1 &= \hat{W}_t^1 - \rho\sigma_S t \\ \breve{W}_t^2 &= \hat{W}_t^2 - \sqrt{1-\rho^2}\sigma_S t\end{aligned} \tag{A.33}$$

Thus:

$$
\begin{aligned}
d\left[\frac{P}{S}\right] &= Pd\left[\frac{1}{S}\right] + \frac{1}{S}dP + d < P, \frac{1}{S} > \\
\frac{d\left[\frac{P}{S}\right]}{\frac{P}{S}} &= \frac{dP}{P} + Sd\left[\frac{1}{S}\right] + \text{termes en dt}
\end{aligned}
\tag{A.34}
$$

By Girsanov theorem, diffusion coefficient don't change and prices in the new numeraire S are Q_S martingales:

$$
\frac{d\left[\frac{P}{S}\right]}{\frac{P}{S}} = -[\sigma_P(t,T) + \rho\sigma_S]d\breve{W}_u^1 - \sigma_S\sqrt{1-\rho^2}d\breve{W}_u^2 \tag{A.35}
$$

Then, a change of time unit gives:

$$
\tau(t) = \int_0^t [\sigma_P^2(u,T) + 2\sigma_S\rho\sigma_P(u,T) + \sigma_S^2]du \tag{A.36}
$$

Thus:

$$
\frac{P(t,T)}{S(t)} = \frac{P(0,T)}{S(0)} \exp\left(B(\tau(t)) - \frac{1}{2}\tau(t)\right) \tag{A.37}
$$

Which leads to:

$$
\frac{1}{S(T)} = \frac{P(0,T)}{S(0)} \exp\left(B\tau(T) - \frac{1}{2}\tau(T)\right) \tag{A.38}
$$

And finally:

$$
\begin{aligned}
Q_S(S_T \geq K) &= Q_S\left(\frac{1}{S_T} \leq \frac{1}{K}\right) \\
&= Q_S\left(\epsilon \leq \frac{ln\left(\frac{S_0}{P(0,T)K}\right) + \frac{1}{2}\tau(T)}{\sqrt{\tau(T)}}\right) \\
&= N(d_1)
\end{aligned}
\tag{A.39}
$$

The computation is similar for B:

$$
\boxed{
\begin{aligned}
C(0,T,K) &= S_0N(d_1) - KP(0,T)N(d_2), \\
d_1 &= \frac{ln\left(\frac{S_0}{P(0,T)K}\right) + \frac{1}{2}\tau(T)}{\sqrt{\tau(T)}}, \\
d_2 &= d_1 - \sqrt{\tau(T)}
\end{aligned}
}
\tag{A.40}
$$

A.4.2 European Put on stock

The Call-Put parity applies for european options:

$$
S + Put = Call + KP(0,T) \tag{A.41}
$$

The put price is directly derived:

$$
\begin{aligned}
Put(0,T,K) &= Call(0,T,K) - S_0 + KP(0,T) \\
&= S_0(N(d_1) - 1) + KP(0,T)(1 - N(d_2)) \\
&= KP(0,T)N(-d_2) - S_0N(-d_1)
\end{aligned}
\tag{A.42}
$$

A.4.3 Vasicek extended model application

Direct application of previous formula gives:

$$\tau(T) = \int_0^T \sigma^2 K^2(T-u) + 2\rho\sigma_S\sigma K(T-u) + \sigma_S^2)du \tag{A.43}$$

Which leads for the ESG:

$$\boxed{\tau(T) = \left(\frac{\sigma^2}{k^2} + \frac{2\rho\sigma\sigma_S}{k} + \sigma_S^2\right)T + 2\left(\frac{\sigma^2}{k^2} - \frac{\rho\sigma\sigma_S}{k}\right)K(T) - \frac{L(T)}{K^2}} \tag{A.44}$$